CALCULUS FOR ENGINEERS

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THE

CALCULUS FOR ENGINEERS

BY

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PREFACE.

THIS book describes what has for many years been the most important part of the regular course in the Calculus for Mechanical and Electrical Engineering students at the Finsbury Technical College. It was supplemented by easy work involving Fourier, Spherical Harmonic, and Bessel Functions which I have been afraid to describe here because the book is already much larger than I thought it would become.

The students in October knew only the most elementary mathematics, many of them did not know the Binomial Theorem, or the definition of the sine of an angle. In July they had not only done the work of this book, but their knowledge was of a practical kind, ready for use in any such engineering problems as I give here.

One such student, Mr Norman Endacott, has corrected the manuscript and proofs. He has worked out many of the exercises in the third chapter twice over. I thank him here for the care he has taken, and I take leave also to say that a system which has, year by year, produced many men with his kind of knowledge of mathematics has a good deal to recommend it. I say this through no vanity but because I wish to encourage the earnest student. Besides I cannot claim more than a portion of the credit, for I do not think that there ever before was such a complete

harmony in the working of all the departments of an educational institution in lectures and in tutorial, laboratory, drawing office and other practical work as exists in the Finsbury Technical College, all tending to the same end; to give an engineer such a perfect acquaintance with his mental tools that he actually uses these tools in his business.

Professor Willis has been kind enough to read through the proofs and I therefore feel doubly sure that no important mistake has been made anywhere.

An experienced friend thinks that I might with advantage have given many more illustrations of the use of squared paper just at the beginning. This is quite possible, but if a student follows my instructions he will furnish all this sort of illustration very much better for himself. Again I might have inserted many easy illustrations of integration by numerical work such as the exercises on the Bull Engine and on Beams and Arches which are to be found in my book on Applied Mechanics. I can only say that I encourage students to find illustrations of this kind for themselves; and surely there must be some limit to spoon feeding.

JOHN PERRY.

ROYAL COLLEGE OF SCIENCE,
LONDON,
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CALCULUS FOR ENGINEERS.

INTRODUCTORY.

1. THE Engineer has usually no time for a general mathematical training—more's the pity—and those young engineers who have had such a training do not always find their mathematics helpful in their profession. Such men will, I hope, find this book useful, if they can only get over the notion that because it is elementary, they know already all that it can teach.

But I write more particularly for readers who have had very little mathematical training and who are willing to work very hard to find out how the calculus is applied in Engineering problems. I assume that a good engineer needs to know only fundamental principles, but that he needs to know these very well indeed.

2. My reader is supposed to have an elementary knowledge of Mechanics, and if he means to take up the Electrical problems he is supposed to have an elementary knowledge of Electrical matters. A common-sense knowledge of the few fundamental facts is what is required; this knowledge is seldom acquired by mere reading or listening to lectures; one needs to make simple experiments and to work easy numerical exercises.

In Mechanics, I should like to think that the mechanical engineers who read this book know what is given in the elementary parts of my books on Applied Mechanics and the Steam and Gas Engine. That is, I assume that they know

the elementary facts about Bending Moment in beams, Work done by forces and the Efficiency of heat engines. Possibly the book may cause them to seek for such knowledge. I take almost all my examples from Engineering, and a man who works these easy examples will find that he knows most of what is called the theory of engineering.

3. I know men who have passed advanced examinations in Mathematics who are very shy, in practical work, of the common formulae used in Engineers' pocket-books. However good a mathematician a student thinks himself to be, he ought to practise working out numerical values, to find for example the value of a^b by means of a table of logarithms, when a and b are any numbers whatsoever. Thus to find $\sqrt[3]{\cdot 014}$, to find $2{\cdot}365^{-0{\cdot}26}$, &c., to take any formula from a pocket-book and use it. He must not only think he knows; he must really do the numerical work. He must know that if a distance 2·454 has been measured and if one is not sure about the last figure, it is rather stupid in multiplying or dividing by this number to get out an answer with many significant figures, or to say that the indicated power of an engine is 324·65 Horse power, when the indicator may be in error 5 per cent. or more. He must know the quick way of finding $3{\cdot}216 \times 4571$ to four significant figures without using logarithms. He ought to test the approximate rule

$$(1+\alpha)^n = 1 + n\alpha,$$

or

$$(1+\alpha)^n (1+\beta)^m = 1 + n\alpha + m\beta,$$

if α and β are small, and see for himself when $\alpha = \cdot 01$ or $-\cdot 01$, or $\beta = \pm \cdot 025$ and $n = 2$ or $\frac{1}{2}$ or $-1\frac{1}{3}$, and $m = 4$ or 2 or -2 or $\frac{1}{3}$ or any other numbers, what errors are involved in the assumption.

† As to Trigonometry, the definitions must be known. For example, Draw BAC an angle of, say, 35°. Take any point B and drop the perpendicular. Measure AB and BC and AC as accurately as possible. Is $AC^2 + BC^2 = AB^2$? Work this out numerically. Now $\frac{BC}{AB} = \sin 35°$, $\frac{AC}{AB} = \cos 35°$, $\frac{BC}{AC} = \tan 35°$. Try if the answers are those given in the tables. Learn how we calculate the other sides of the

triangle ABC when we know one side and one of the acute angles. Learn also that the sine of 130° is positive, and the cosine of 130° is negative. Also try with the book of tables if

$$\sin(A+B) = \sin A \,.\, \cos B + \cos A \,.\, \sin B,$$

where A and B are any two angles you choose to take. There are three other rules like this. In like manner the four which we obtain by adding these formulae and subtracting them, of which this is one,

$$2 \sin \alpha \,.\, \cos \beta = \sin(\alpha + \beta) + \sin(\alpha - \beta);$$

also $$\cos 2A = 1 - 2 \sin^2 A = 2 \cos^2 A - 1.$$

Before readers have gone far in this book I hope they will be induced to take up the useful (that is, the elementary and interesting) part of trigonometry, and prove all rules for themselves, if they haven't done so already.

Calculate an angle of 1·6 degrees in radians (1 radian is equal to 57·296 degrees); see how much the sine and tangent of this angle differ from the angle itself. Remember that when in mathematics we say sin x, x is supposed to be in radians.

I do not expect a man to know much about advanced algebra, but he is supposed to be able to give the factors of $x^2 + 7x + 12$ or of $x^2 - a^2$ for example; to be able to simplify expressions. It is not a knowledge of permutations or combinations or of the theory of equations, of Geometrical Conics or tangent planes to quadrics, that the Engineer wants. Happy is the Engineer who is also a mathematician, but it is given to only a few men to have the two so very different powers.

A prolonged experience of workshops, engineers and students has convinced me that although a Civil Engineer for the purposes of surveying may need to understand the solution of triangles, this and many other parts of the Engineer's usual mathematical training are really useless to the mechanical or electrical engineer. This sounds unorthodox, but I venture to emphasise it. The young engineer cannot be drilled too much in the mere simplification of algebraic and trigonometrical expressions, including expressions involving $\sqrt{-1}$, and the best service done by

elementary calculus work is in inducing students to again undergo this drilling.

But the engineer needs no artificial mental gymnastics such as is furnished by Geometrical Conics, or the usual examination-paper puzzles, or by evasions of the Calculus through infinite worry with elementary Mathematics. The result of a false system of training is seen in this, that not one good engineer in a hundred believes in what is usually called theory.

4. I assume that every one of my readers is thoroughly well acquainted already with the fundamental notion of the Calculus, only he doesn't know it in the algebraic form. He has a perfect knowledge of *a rate*, but he has never been accustomed to write $\frac{dy}{dx}$; he has a perfect knowledge of an area, but he has not yet learnt the symbol used by us, $\int f(x).dx$. He has the idea, but he does not express his idea in this form.

I assume that some of my readers have passed difficult examinations in the Calculus, that they can differentiate any function of x and integrate many; that they know how to work all sorts of difficult exercises about Pedal Curves and Roulettes and Elliptic Integrals, and to them also I hope to be of use. Their difficulty is this, their mathematical knowledge seems to be of no use to them in practical engineering problems. Give to their x's and y's a physical meaning, or use p's and v's instead, and what was the easiest book exercise becomes a difficult problem. I know such men who hurriedly skip in reading a book when they see a $\frac{dp}{dt}$, or a sign of integration.

5. When I started to write this book I thought to put the subject before my readers as I have been able, I think—I have been told—very successfully, to bring it before some classes of evening students; but much may be done in lectures which one is unable to do in a cold-blooded fashion sitting at a table. One misses the intelligent eyes of an audience, warning one that a little more explanation is needed

or that an important idea has already been grasped. An idea could be given in the mere drawing of a curve and illustrations chosen from objects around the lecture-room.

Let the reader skip judiciously; let him work up no problem here in which he has no professional interest. The problems are many, and the best training comes from the careful study of only a few of them.

The reader is expected to turn back often to read again the early parts.

The book would be unwieldy if I included any but the more interesting and illustrative of engineering problems. I put off for a future occasion what would perhaps to many students be a more interesting part of my subject, namely, illustrations from Engineering (sometimes called Applied Physics) of the solution of Partial Differential Equations. Many people think the subject one which cannot be taught in an elementary fashion, but Lord Kelvin showed me long ago that there is no useful mathematical weapon which an engineer may not learn to use. A man learns to use the Calculus as he learns to use the chisel or the file on actual concrete bits of work, and it is on this idea that I act in teaching the use of the Calculus to Engineers.

This book is not meant to supersede the more orthodox treatises, it is rather an introduction to them. In the first chapter of 160 pages, I do not attempt to differentiate or integrate any function of x, except x^n. In the second chapter I deal with e^{ax}, and $\sin(ax + c)$. The third chapter is more difficult.

For the sake of the training in elementary Algebraic work, as much as for use in Engineering problems, I have included a set of exercises on general differentiation and integration.

Parts in smaller type, and the notes, may be found too difficult by some students in a first reading of the book. An occasional exercise may need a little more knowledge than the student already possesses. His remedy is to skip.

CHAPTER I.

x^n.

6. EVERYBODY has already the notions of **Co-ordinate Geometry** and uses **squared paper.** Squared paper may be bought at sevenpence a quire: people who are ignorant of this fact and who pay sevenpence or fourteen pence a sheet for it must have too great an idea of its value to use it properly.

When a **merchant** has in his office a sheet of squared paper with points lying in a curve which he adds to day by day, each point showing the price of iron, or copper, or cotton yarn or silk, at any date, he is using Co-ordinate Geometry. Now to what uses does he put such a curve? 1. At any date he sees what the price was. 2. He sees by the *slope* of his curve the *rate* of increase or fall of the price. 3. If he plots other things on the same sheet of paper at the same dates he will note what effect their rise and fall have upon the price of his material, and this may enable him to prophesy and so make money. 4. Examination of his curve for the past will enable him to prophesy with more certainty than a man can do who has no records.

Observe that **any point represents two things;** its horizontal distance from some standard line or axis is called one co-ordinate, we generally call it the x co-ordinate and it is measured horizontally to the right of the axis of y; some people call it the *abscissa*; this represents time in his case. The other co-ordinate (we usually call it the y co-ordinate or *the ordinate*, simply), the vertical distance of the point above some standard line or axis; this represents his price. In the newspaper you will find curves showing how the thermometer and barometer are rising and falling. I once read a clever article upon the way in which the English population and wealth and taxes were increasing; the reasoning was very

difficult to follow. On taking the author's figures however and plotting them on squared paper, every result which he had laboured so much to bring out was plain upon the curves, so that a boy could understand them. Possibly this is the reason why some writers do not publish curves: if they did, there would be little need for writing.

7. A man making **experiments** is usually finding out how one thing which I shall call y depends upon some other thing which I shall call x. Thus the pressure p of saturated steam (water and steam present in a vessel but no air or other fluid) is always the same for the same temperature. A curve drawn on squared paper enables us for any given temperature to find the pressure or vice versa, but it shows the rate at which one increases relatively to the increase of the other and much else. I do not say that the curve is always better than the table of values for giving information; some information is better given by the curve, some by the table. Observe that when we represent any quantity by the length of a line we represent it to some scale or other; 1 inch represents 10 lbs. per square inch or 20 degrees centigrade or something else; it is always to *scale* and according to a convention of some kind, for of course a distance 1 inch is a very different thing from 20 degrees centigrade.

When one has two columns of observed numbers to plot on squared paper one does it, 1. To see if the points lie in any regular curve. If so, the simpler the curve the simpler is the law that we are likely to find. 2. To correct errors of observation. For if the points lie nearly in a simple regular curve, if we draw the curve that lies most evenly among the points, using thin battens of wood, say, then it may be taken as probable that if there were no errors of observation the points would lie exactly in such a curve. Note that when a point is -5 feet to the right of a line, we mean that it is 5 feet to the left of the line. I have learnt by long experience that it is worth while to spend a good deal of time subtracting from and multiplying one's quantities to fit the numbers of squares (so that the whole of a sheet of paper is needed for the points) before beginning to plot.

Now let the reader buy some squared paper and without asking help from anyone let him plot the results of some

observations. Let him take for example a Whitaker's Almanack and plot from it some sets of numbers; the average temperature of every month last year; the National Debt since 1688; the present value of a lease at 4 per cent. for any number of years; the capital invested in Railways since 1849; anything will do, but he had better take things in which he is interested. If he has made laboratory observations he will have an absorbing interest in seeing what sort of law the squared paper gives him.

8. As the observations may be on pressure p and temperature t, or p and volume v, or v and t, or Indicated Horse Power and Useful Horse Power of a steam or gas engine, or amperes and volts in electricity, and we want to talk generally of any such pair of quantities, I shall use x and y instead of the p's and v's and t's and all sorts of letters. The short way of saying that there is some law connecting two variable quantities x and y is $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$...(1), or in words "there is some equation connecting x and y." Any expression which contains x and y (it may contain many other letters and numbers also) is said to be a *function* of x and y and we use such symbols as $F(x, y)$, $f(x, y)$, $Q(x, y)$ etc. to represent functions in general when we don't know what the expressions really are, and often when we do know, but want to write things shortly. Again we use $F(x)$ or $f(x)$ or any other convenient symbol to mean "any mathematical expression containing x," and we say "let $f(x)$ be any function of x." Thus $y = f(x)$...(2) stands for any equation which would enable us when given x to calculate y.

The law $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$ comes under the form (1) given above, whereas if we calculate y in terms of x and get $y = \pm \frac{4}{5}\sqrt{25 - x^2}$ we have the form (2). But in either case we have the same law connecting y and x. In pure mathematics x and y are actual distances; in applied mathematics x and y stand for the quantities which we are comparing and which are represented to scale.

9. **'Graph' Exercises.**

I. Draw the curve $y = 2 + \frac{1}{30}x^2$.

Take $x = 0$ and we find $y = 2$; take $x = 1$, then $y = 2{\cdot}0333$;

take $x=2$, then $y=2+\cdot 1333=2\cdot 133$; and so on. Now plot these values of x and y on your sheet of squared paper. The curve is a parabola.

II. Draw the curve $y=2-\frac{1}{5}x+\frac{1}{30}x^2$ which is also a parabola, in the same way, on the same sheet of paper.

III. Draw the curve $xy=120$. Now if $x=1$, $y=120$; if $x=2$, $y=60$; if $x=3$, $y=40$; if $x=4$, $y=30$ and so on: this curve is a rectangular hyperbola.

IV. Draw $yx^{1\cdot 414}=100$ or $y=100x^{-1\cdot 414}$. If the student cannot calculate y for any value of x, he does not know how to use logarithms and the sooner he does know how to use logarithms the better.

V. Draw $y=ax^n$ where a is any convenient number. I advise the student to spend a lot of time in drawing members of this great family of useful curves. Let him try $n=-1$ (he drew this in III. above), $n=-2$, $n=-1\frac{1}{2}$, $n=-\frac{1}{2}$, $n=-0\cdot 1$, $n=0$, $n=\frac{1}{2}$, $n=\frac{3}{4}$, $n=1$, $n=1\frac{1}{2}$, $n=2$ (this is No. I. above), $n=3$, $n=4$ &c.

VI. Draw $\mathbf{y=a\sin(bx+c)}$ taking any convenient numbers for a, b and c.

Advice. As $bx+c$ is in *radians* (one radian is $57\cdot 2958$ degrees) and the books of tables usually give angles in degrees, choose numbers for b and c which will make the arithmetical work easy. Thus take $b=1\div 114\cdot 6$, take c the number of radians which correspond to say 30°

$$\left(\text{this is } \frac{\pi}{6} \text{ or } \cdot 5236\right).$$

Let $a=5$ say. Now let $x=0, 10, 20$, &c., and calculate y.

Thus when $x=6$, $y=5\sin\left(\dfrac{6}{114\cdot 6}+\cdot 5236\right)$; but if the angle is converted into *degrees* we have

$$y=5\sin\left(\tfrac{1}{2}6+30\text{ degrees}\right)=5\sin 33^\circ=2\cdot 723.$$

Having drawn the above curve, notice what change would occur if c were changed to 0 or $\dfrac{\pi}{4}$ or $\dfrac{\pi}{3}$ or $\dfrac{\pi}{2}$. Again, if a were changed. More than a week may be spent on this curve, very profitably.

VII. Draw $\mathbf{y = ae^{bx}}$. Try $b=1$ and $a=1$; try other values of a and b; take at least two cases of negative values for b.

In the above work, get as little help from teachers as possible, but help from fellow students will be very useful especially if it leads to wrangling about the subject.

The reason why I have dwelt upon the above seven cases is this:—Students learn usually to differentiate and integrate the most complicated expressions: but when the very simplest of these expressions comes before them in a real engineering problem they fight shy of it. Now it is very seldom that an engineer ever has to face a problem, even in the most intricate part of his theoretical work, which involves a knowledge of more functions than these three

$$y = ax^n, \quad y = a\sin(bx+c), \quad y = ae^{bx},$$

but these three must be thoroughly well understood and the engineering student must look upon the study of them as his most important theoretical work.

Attending to the above three kinds of expression is a student's real business. I see no reason, however, for his not having a little amusement also, so he may draw the curves

$$x^2 + y^2 = 25 \text{ (Circle)}, \quad \frac{x^2}{25} + \frac{y^2}{16} = 1 \text{ (Ellipse)},$$

$$\frac{x^2}{25} - \frac{y^2}{16} = 1 \text{ (Hyperbola)},$$

and some others mentioned in Chapter III., but from the engineer's point of view these curves are comparatively uninteresting.

10. Having studied $y = e^{-ax}$ and $y = b\sin(cx+g)$ a student will find that he can now easily understand one of the most important curves in engineering, viz:

$$\mathbf{y = be^{-ax}\sin(cx+g)}.$$

He ought first to take such a curve as has already been studied by him, $y = b\sin(cx+g)$; plot on the same sheet of paper $y = e^{-ax}$; and multiply together the ordinates of the two curves at many values of x to find the ordinate of the

new curve. The curve is evidently wavy, y reaching maximum and minimum values; y represents the displacement of a pendulum bob or pointer of some measuring instrument whose motion is damped by fluid or other such friction, x being the time, and a student will understand the curve much better if he makes observations of such a motion, for example with a disc of lead immersed in oil vibrating so slowly under the action of torsional forces in a wire that many observations of its angular position (using pointer and scale of degrees) which is called y, x being the time, may be made in one swing. The distance or angle from an extreme position on one side of the zero to the next extreme position on the other side is called the length of one swing. The Napierian logarithm of the ratio of the length of one swing to the next or one tenth of the logarithm of the ratio of the first swing to the eleventh is evidently a multiplied by half the periodic time, or it is a multiplied by the time occupied in *one swing*. This **logarithmic decrement** as it is called, is rather important in some kinds of measurement.

11. When by means of a drawing or a model we are able to find the **path of any point** and where it is in its path when we know the position of some other point, we are always able to get the same information algebraically.

Example (1). A point F and a straight line DD being given; what is the path of a point P when it moves so that its distance from the point F is always in the same ratio to its distance from the straight line?

Thus in the figure let $PF = e \times PD$ ... (1), where e is a constant. Draw EFX at right angles to DD. If the distance PD is called x and the perpendicular PG is y; our problem is this;—What is the equation connecting x and y? Now all we have to do is to express (1) in terms of x and y. Let EF be called a.

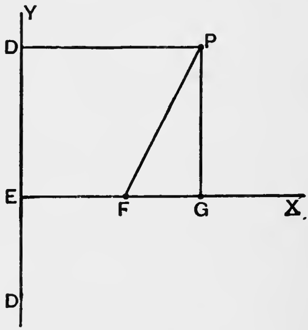

Fig. 1.

Thus

$$PF = \sqrt{PG^2 + FG^2} = \sqrt{y^2 + (x-a)^2}$$

so that, squaring (1) we have

$$y^2 + (x-a)^2 = e^2x^2 \text{ .... (2).}$$

This is the answer. If e is 1 the curve is called a **parabola.** If e is greater than 1, the curve is called an **hyperbola.** If e is less than 1, the curve is called an **ellipse.**

Example (2). The circle APQ rolls on the straight line OX. What is the Path of any point P on the circumference? If when P

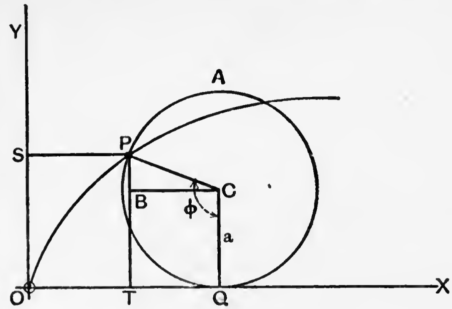

Fig. 2.

touched the line it was at O, let OX and OY be the axes, and let SP be x and PT be y. Let the radius of the circle be a. Let the angle PCQ be called ϕ. Draw CB, perpendicular to PT. Observe that

$$PB = a \,.\, \sin PCB = a \sin(\phi - 90) = -a \cos\phi,$$

$$BC = a \cos PCB = a \sin\phi.$$

Now the arc $QP = a \,.\, \phi = OQ$. Hence as $x = OQ - BC$, and

$$y = BT + PB,$$

we have

$$\left.\begin{aligned} x &= a\phi - a\sin\phi \\ y &= a \quad - a\cos\phi \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3).$$

If from (3) we eliminate ϕ we get one equation connecting x and y. But it is better to retain ϕ and to have two equations because of the greater simplicity of calculation. In fact the two equations (3) may be called the equation to the curve. The curve is called the **cycloid** as all my readers know already.

Example (3). **A crank and connecting rod** work a slider in a straight path. Where is the slider for any position of the crank?

Let the path be in the direction of the centre of the crank shaft.

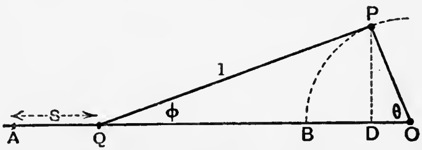

Fig. 3.

If A is the end of the path, evidently AO is equal to $l+r$, r being length of crank.

It is well to remember in all such problems that if we project all the sides of a closed figure upon any two straight lines, we get two independent equations. Projecting on the horizontal we see that

$$\left.\begin{array}{r} s+l\cos\phi+r\cos\theta=l+r \\ \text{Projecting on the vertical} \qquad l\sin\phi=r\sin\theta \end{array}\right\}\text{.................(1).}$$

If we eliminate ϕ from these equations we can calculate s for any value of θ. The student ought to do this for himself, but I am weak enough to do it here. We see that from the second equation of (1)

$$\cos\phi=\sqrt{1-\frac{r^2}{l^2}\sin^2\theta},$$

so that the first becomes

$$s=l\left\{1-\sqrt{1-\frac{r^2}{l^2}\sin^2\theta}\right\}+r(1-\cos\theta)\text{...* ....(2).}$$

Students ought to work a few exercises, such as;—1. The ends A and B of a rod are guided by two straight slots OA and OB which are at right angles to one another; find the equation to the path of any point P in the rod. 2. In **Watt's parallel motion** there is a point which moves nearly in a straight path. Find the equation to its complete path.

In fig. 4 the Mean Position is shown as $OABC$. The best place for P is such that $BP/PA=OA/CB$. Draw the links in any other

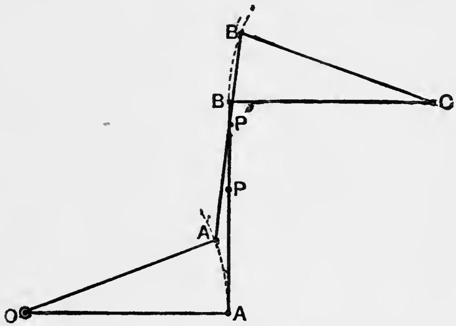

Fig. 4.

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* Note that if as is usual, $\frac{r^2}{l^2}$ is a small fraction, then since $\sqrt{1-a}=1-\frac{1}{2}a$ when a is small, we can get an approximation to the value of s, which can be expressed in terms of θ and 2θ. This is of far more importance than it here seems to be. When the straight path of Q makes an angle a with the line joining its middle point and O, if a is not large, it is evident that s is much the same as before, only divided by cos a. When a is large, the algebraic expression for s is rather complicated, but good approximations may always be found which will save trouble in calculation.

position. The complete path of P' would be a figure of 8. 3. Find the equation to the path of a point in the middle of an ordinary connecting rod. 4. A, the end of a link, moves in a straight path COC', O being the middle of the path, with a simple harmonic motion $OA = a \sin pt$, where t is time; the other end B moves in a straight path OBD which is in a direction at right angles to COC'; what is B's motion? Show that it is approximately a simple harmonic motion of twice the frequency of A. 5. In any slide valve gear, in which there are several links, &c. driven from a uniformly rotating crank; note this fact, that the motion of any point of any link in any particular direction consists of a fundamental simple harmonic motion of the same frequency as the crank, together with an octave. The proper study of **Link Motions and Radial valve gears** from this point of view is worth months of one's life, for this contains the secret of why one valve motion gives a better diagram than another. Consider for example the Hackworth gear with a curved and with a straight slot. What is the difference? See Art. 122.

12. Plotted points lying in a straight line. Proofs will come later; at first the student ought to get well acquainted with the thing to be proved. I have known boys able to *prove* mathematical propositions who did not really know what they had proved till years afterwards.

Take any expression like $y = a + bx$, where a and b are numbers. Thus let $y = 2 + 1\frac{1}{2}x$. Now take $x = 0$, $x = 1$, $x = 2$, $x = 3$, &c. and in each case calculate the corresponding value of y. Plot the corresponding values of x and y as the co-ordinates of points on squared paper. You will find that they lie exactly in a straight line. Now take say $y = 2 + 3x$ or $2 + \frac{1}{2}x$ or $2 - \frac{1}{2}x$ or $2 - 3x$ and you will find in every case a straight line. Men who think they know a little about this subject already will not care to take the trouble and if you do not find yourselves interested, I advise you not to take the trouble either; yet I know that it is worth your while to take the trouble. Just notice that in every case I have given you the same value of a and consequently all your lines have some one thing in common. What is it? Take this hint, a is the value of y when $x = 0$.

Again, try $y = 2 + 1\frac{1}{2}x$, $\quad y = 1 + 1\frac{1}{2}x$, $y = 0 + 1\frac{1}{2}x$,

$$y = -1 + 1\tfrac{1}{2}x, \qquad y = -2 + 1\tfrac{1}{2}x,$$

and so see what it means when b is the same in every case. You will find that all the lines with the same b have the

same slope and indeed I am in the habit of calling b the *slope* of the line.

If $y = a + bx$, when $x = x_1$, find y and call it y_1,
when $x = x_1 + 1$, find y and call it y_2.

It is easy to show that $y_2 - y_1 = b$. So that what I mean by the slope of a straight line is its rise for a horizontal distance 1. (Note that when we say that a road rises $\frac{1}{20}$ or 1 in 20, we mean 1 foot rise for 20 feet along the sloping road. Thus $\frac{1}{20}$ is the sine of the angle of inclination of the road to the horizontal; whereas our *slope* is measured in a different way). Our slope is evidently the tangent of the inclination of the line to the horizontal. Looking upon y as a quantity whose value depends upon that of x, observe that *the rate of increase of y relatively to the increase of x* is constant, being indeed b, the slope of the line. The symbol used for this rate is $\dfrac{dy}{dx}$. Observe that it is one symbol; it does not mean $\dfrac{d \times y}{d \times x}$. Try to recollect the statement that if $y = a + bx$, $\dfrac{dy}{dx} = b$, and that if $\dfrac{dy}{dx} = b$, then it follows that $y = A + bx$, where A is some constant or other.

Any equation of the first degree connecting x and y such as $Ax + By = C$ where A, B and C are constants, can be put into the shape $y = \dfrac{C}{B} - \dfrac{A}{B}x$, so that it is the equation to a straight line whose slope is $-\dfrac{A}{B}$ and which passes through the point whose $x = 0$, whose $y = \dfrac{C}{B}$, called point $\left(0, \dfrac{C}{B}\right)$. Thus $4x + 2y = 5$ passes through the point $x = 0$, $y = 2\frac{1}{2}$ and its slope is -2. That is, y diminishes as x increases. You are expected to draw this line $y = 2\frac{1}{2} - 2x$ and distinguish the difference between it and the line $y = 2\frac{1}{2} + 2x$. Note what is meant by positive and what by negative slope. Draw a few curves and judge approximately by eye of the slope at a number of places.

13. Problems on the straight line.

1. Given the slope of a straight line; if you are also

told that it passes through the point whose $x=3$, and whose $y=2$, what is the equation to the line? Let the slope be 0·35.

The equation is $y=a+0{\cdot}35x$. where a is not known.

But (3, 2) is a point on the line, so that $2=a+0{\cdot}35\times 3$, or $a=0{\cdot}95$ and hence the line is $y=0{\cdot}95+0{\cdot}35x$.

2. What is the slope of any line at right angles to $y=a+bx$? Let AB be the given line, cutting OX in C. Then

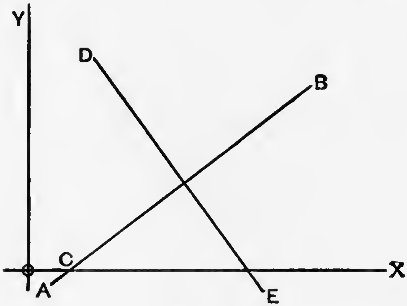

Fig. 5.

$b=\tan BCX$. If DE is any line at right angles to the first, its slope is $\tan DEX$ or $-\tan DEC$ or $-\cot BCE$ or $-\frac{1}{b}$.

So that $y=A-\frac{1}{b}x$ is typical of all lines at right angles to $y=a+bx$; A being any constant.

3. Where do the two straight lines $Ax+By+C=0$ and $Mx+Ny+S=0$ meet? Answer, In the point whose x and y satisfy both the equations. We have therefore to do what is done in Elementary Algebra, solve simultaneous equations.

4. When $\tan\alpha$ and $\tan\beta$ are known, it is easy to find $\tan(\alpha-\beta)$, and hence when the straight lines $y=a+bx$ and $y=m+nx$ are given, it is easy to find the angle between them.

5. The line $y=a+bx$ passes through the points $x=1$, $y=2$, and $x=3$, $y=1$, find a and b.

6. A line $y=a+bx$ is at right angles to $y=2+3x$ and passes through the point $x=1$, $y=1$. Find a and b.

14. Obtaining Empirical Formulae.

When in the laboratory we have made measurements of two quantities which depend upon one another, we have a table showing corresponding values of the two, and we wish to see if there is a simple relation between them, we plot the values to convenient scales as the co-ordinates of points on squared paper. If some regular curve (a curve without singular points as I shall afterwards call it) seems as if it might pass through all the points, save for possible errors of measurement, we try to obtain a formula $y=f(x)$, which we may call the law or rule connecting the quantities called y and x.

If the points appear as if they might lie on a straight line, a stretched thread may be used to help in finding its most probable position. There is a tedious algebraic method of finding the straight line which represents the positions of the points with least error, but for most engineering purposes the stretched string method is sufficiently accurate.

If the curve seems to follow such a law as $y=a+bx^2$, plot y and the square of the observed measurement, which we call x, as the co-ordinates of points, and see if they lie on a straight line. If the curve seems to follow such a law as $\mathbf{y}=\dfrac{\mathbf{ax}}{\mathbf{1+bx}}$ ......(1), which is the same as $\dfrac{y}{x}+by=a$, divide each of the quantities which you call y by the corresponding quantity x; call the ratio X. Now plot the values of X and of y on squared paper; if a straight line passes through the plotted points, then we have such a law as $X=A+By$, or $\dfrac{y}{x}=A+By$, or $y=\dfrac{Ax}{1-Bx}$, so that (1) is true.

Usually we can apply the stretched thread method to find the probability of truth of any law containing only two constants.

Thus, suppose measurements to be taken from the expansion part of a gas engine indicator diagram. It is important for many purposes to obtain an empirical formula connecting p and v, the pressure and volume. I always find that the following rule holds with a fair amount of accuracy $\mathbf{pv^s=C}$ where s and C are two constants. We do not much care to know C, but if there is such a rule, the value of s is very important*. To test if this rule holds, plot $\log p$ and $\log v$ as the co-

* There is no known physical reason for expecting such a rule to hold. At first I thought that perhaps most curves drawn at random approximately like hyperbolas would approximately submit to such a law as $yx^n=C$, but I found that this was by no means the case. The following fact is worth mentioning. When my students find, in carrying out the above rule that $\log p$ and $\log v$ do not lie in a straight line, I find that they have

ordinates of points on squared paper (common logarithms will do). If they lie approximately in a straight line, we see that

$$\log p + s \log v = c$$

a constant, and therefore the rule holds.

When we wish to test with a formula containing three independent constants we can often reduce it to such a shape as

$$Av + Bw + Cz = 1 \quad\text{.................................}\quad (2),$$

where v, w, z contain x and y in some shape. Thus to test if $y = \dfrac{a+bx}{1+cx}$, we have $y + cxy = a + bx$, or $\dfrac{y}{a} + \dfrac{c}{a} xy - \dfrac{b}{a} x = 1$. Here y itself is the old v, xy is the old w, and x itself is the old z.

If (2) holds, and if v, w and z were plotted as the three co-ordinates of a point in space, all the points ought to lie in a plane. By means of three sides of a wooden box and a number of beads on the ends of pointed wires this may be tried directly; immersion in a tank of water to try whether one can get the beads to lie in the plane of the surface of the water, being used to find the plane. I have also used a descriptive geometry method to find the plane, but there is no method yet used by me which compares for simplicity with the stretched thread method in the other case.

But no hard and fast rules can be given for trying all sorts of empirical formulae upon one's observed numbers. The student is warned that his formula is an empirical one, and that he must not deal with it as if he had discovered a natural law of infinite exactness.

When other formulae fail, we try

$$y = a + bx + cx^2 + ex^3 + \&c.,$$

because we know that with sufficient terms this will satisfy any curve. When there are more than two constants, we often find them by a patient application of what is called the method of least squares. To test if the pressure and temperature of saturated steam follow the rule $p = a(\theta + \beta)^n$... (3), where θ is temperature, Centigrade, say, three constants have to be found. The only successful plan tried by me is one in which I guess at β. I know that β is nearly 40. I ask one student to try $\beta = 40$, another to try $\beta = 41$, another $\beta = 39$ and so on;

made a mistake in the amount of clearance. Too much clearance and too little clearance give results which depart in opposite ways from the straight line. It is convenient in many calculations, if there is such an empirical formula, to use it. If not, one has to work with rules which instruct us to draw tangents to the curve. Now it is an excellent exercise to let a number of students trace the same curve with two points marked upon it and to let them all independently draw tangents at those points to their curve, and measure the angle between them. It is extraordinary what very different lines they will draw and what different angles they may obtain. Let them all measure by trial the radius of curvature of the curve at a point; in this the discrepancies are greater than before.

they are asked to find the rule (3) which most accurately represents p and θ between, say $p=7$ lb. per sq. inch, and $p=150$. He who gets a straight line lying most evenly (judging by the eye) among the points, when $\log p$ and $\log(\theta+\beta)$ are used as co-ordinates, has used the best value of β. The method may be refined upon by ingenious students.

(See end of Chap. I.)

15. We have now to remember that if $y=a+bx$, then $\frac{dy}{dx}=b$, and if $\frac{dy}{dx}=b$, then $y=A+bx$, where A is some constant.

Let us prove this algebraically.

If $y=a+bx$. Take a particular value of x and calculate y. Now take a new value of x, call it $x+\delta x$, and calculate the new y, call it $y+\delta y$,

$$y+\delta y=a+b\,(x+\delta x).$$

Subtract $y=a+bx$ and we get

$$\delta y=b\delta x, \text{ or } \frac{\delta y}{\delta x}=b,$$

and, however small δx or δy may become their ratio is b, we therefore say $\frac{dy}{dx}=b$.

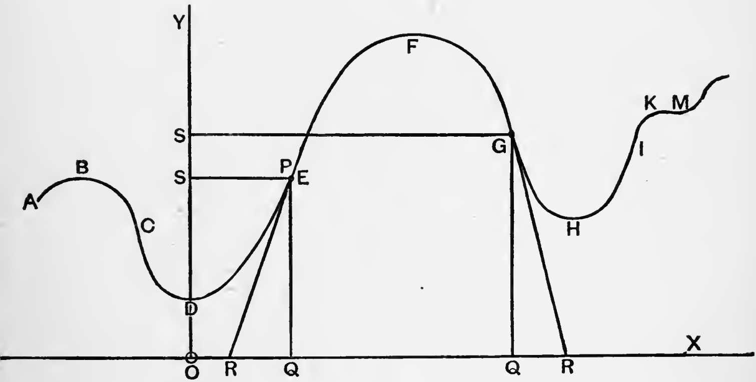

Fig. 6.

16. In the curve of fig. 6 there is **positive slope** (y *increases* as x increases) in the parts AB, DF and HI and

negative slope (y *diminishes* as x increases) in the parts BD and FH. The slope is 0 at B and F which are called points of **maximum** or points where y is a maximum; and it is also 0 at D and H which are points of **minimum**. The point E is one in which the slope ceases to increase and begins to diminish: it is a point of **inflexion**.

Notice that if we want to know the slope at the point P we first choose a point F which is near to P. (Imagine that in fig. 6 the little portion of the curve at P is magnified a thousand times.) Call $PS=x$, $PQ=y$; $NF=x+\delta x$, $FL=y+\delta y$, so that $PM=\delta x$, $FM=\delta y$. Now FM/PM or $\delta y/\delta x$ is the *average* slope between P and F. It is tan FPM. Imagine the same sort of figure drawn but for a point F' nearer to P. Again, another, still nearer P. Observe that the straight line FP or $F'P$ or $F''P$ gets gradually more and more nearly what we mean by the tangent to the curve at P. In every case $\delta y/\delta x$ is the tangent of the angle which the line FP or $F'P$ or $F''P$ makes with the horizontal, and so we see that *in the limit* the slope of the line or dy/dx at P is the tangent of the angle which the tangent at P makes with the axis of X. If then, instead of judging roughly by the eye as we did just now in discussing fig. 6, we wish to measure very accurately the slope at the point P;—Note that the *slope* is independent of where the axis of X is, so long as it is a horizontal line, and I take care in using my rule here given, to draw OX *below* the part of the curve where I am studying the slope. Draw a tangent PR to the curve, cutting OX in R. Then the slope is tan PRX. If drawn and lettered according to my instructions, observe that PRX is always an acute angle when the slope is positive and is always an obtuse angle when the slope is negative.

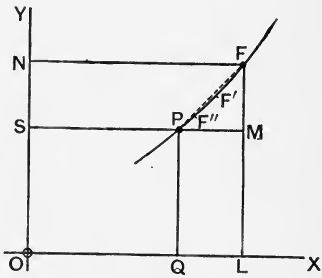

Fig. 7.

Do not forget that the slope of the curve at any point means the rate of increase of y there with regard to x, and

that we may call it the *slope* of the curve or tan PRQ or by the symbol $\frac{dy}{dx}$ or **"the differential coefficient of y with regard to x,"** and all these mean the same thing.

Every one knows what is meant when on going up a hill one says that the **slope** is changing, the slope is diminishing, the slope is increasing; and in this knowledge he already possesses the fundamental idea of the calculus.

17. We all know what is meant when in a railway train we say **"we are going at 30 miles per hour."** Do we mean that we have gone 30 miles in the last hour or that we are really going 30 miles in the next hour? Certainly not. We may have only left the terminus 10 minutes ago; there may be an accident in the next second. What we mean is merely this, that the last distance of 3 miles was traversed in the tenth of an hour, or rather, the last distance of 0·0003 miles was traversed in 0·00001 hour. This is not exactly right; it is not till we take still shorter and shorter distances and divide by the times occupied that we approach the true value of the speed. Thus it is known that a body falls freely vertically through the following distances in the following intervals of time after two seconds from rest, at London. That is between 2 seconds from rest and 2·1 or 2·01 or 2·001, the distances fallen through are given. Each of these divided by the interval of time gives the average velocity during the interval.

Intervals of time in seconds	·1	·01	·001
Distances in feet fallen through	6·601	·6456	·064416
Average velocities	66·01	64·56	64·416

We see that as the interval of time after 2 seconds is taken less and less, the average velocity during the interval approaches more and more the true value of the velocity at 2 seconds from rest which is exactly 64·4 feet per second.

We may find the true velocity at any time when we know the law connecting s and t as follows.

Let $\mathbf{s = 16 \cdot 1 t^2}$, the well known law for bodies falling freely at London. If t is given of any value we can calculate

s. If t has a slightly greater value called $t + \delta t$ (here δt is a symbol for a small portion of time, it is not $\delta \times t$, but a very different thing), and if we call the calculated space $s + \delta s$, then $s + \delta s = 16{\cdot}1\,(t + \delta t)^2$ or $16{\cdot}1\,\{t^2 + 2t\,.\,\delta t + (\delta t)^2\}$. Hence, subtracting, $\delta s = 16{\cdot}1\,\{2t\,.\,\delta t + (\delta t)^2\}$, and this formula will enable us to calculate accurately the space δs passed through between the time t and the time $t + \delta t$. The average velocity during this interval of time is $\delta s \div \delta t$ or

$$\frac{\delta s}{\delta t} = 32{\cdot}2t + 16{\cdot}1\,\delta t.$$

Please notice that this is absolutely correct; there is no vagueness about it.

Now I come to the important idea; as δt gets smaller and smaller, $\dfrac{\delta s}{\delta t}$ approaches more and more nearly $32{\cdot}2t$, the other term $16{\cdot}1\delta t$ becoming smaller and smaller, and hence we say that **in the limit**, $\delta s/\delta t$ is truly $32{\cdot}2t$. The limiting value of $\dfrac{\delta s}{\delta t}$ as δt gets smaller and smaller is called $\dfrac{ds}{dt}$ or the rate of change of s as t increases, or the differential coefficient of s with regard to t, or it is called the velocity at the time t.

Now surely there is no such great difficulty in catching the idea of a limiting value. Some people have the notion that we are stating something that is only approximately true; it is often because their teacher will say such things as "reject $16{\cdot}1\delta t$ because it is small," or "let dt be an infinitely small amount of time" and they proceed to divide something by it, showing that although they may reach the age of Methuselah they will never have the common sense of an engineer.

Another trouble is introduced by people saying "let $\delta t = 0$ and $\dfrac{\delta s}{\delta t}$ or $\dfrac{ds}{dt}$ is so and so." The true statement is, "as δt gets smaller and smaller without limit, $\dfrac{\delta s}{\delta t}$ approaches more and more nearly the finite value $32{\cdot}2t$," and as I have already said, everybody uses the important idea of a limit every day of his life.

From the law connecting s and t, if we find $\frac{ds}{dt}$ or the velocity, we are said to **differentiate** s with regard to the time t. When we are given ds/dt and we reverse the above process we are said to **integrate.**

If I were lecturing I might dwell longer upon the correctness of the notion of a rate that one already has, and by making many sketches illustrate my meaning. But one may listen intently to a lecture which seems dull enough in a book. I will, therefore, make a virtue of necessity and say that my readers can illustrate my meaning perfectly well to themselves if they do a little thinking about it. After all my great aim is to make them less afraid than they used to be of such symbols as dy/dx and $\int y \, . \, dx$.

18. Given s and t in any kind of motion, as a set of numbers. How do we study the motion? For example, imagine a Bradshaw's Railway Guide which not merely gave a few stations, but some hundred places between Euston and Rugby. The entries might be like this: s would be in miles, t in hours and minutes. $s = 0$ would mean Euston.

s	t
0	10 o'clock
3	10..10
5	10..15
7	10..20
7	10..23
9	10..28
12	10..33
	&c.

One method is this: plot t (take times after 10 o'clock) horizontally and s vertically on a sheet of squared paper and draw a curve through the points.

The *slope* of this curve at any place represents the velocity of the train to some scale which depends upon the scales for s and t.

Observe places where the velocity is great or small. Between $t=10.20$ and $t=10.23$ observe that the velocity is 0. Indeed the train has probably stopped altogether. To be absolutely certain, it would be necessary to give s for every value of t, and not merely for a few values. A curve alone can show every value. I do not say that the table may not be more valuable than the curve for a great many purposes.

If the train stopped at any place and travelled towards Euston again, we should have negative slope to our curve and negative velocity.

Note that **acceleration** being rate of change of velocity with time, is indicated by the rate of change of the slope of the curve. Why not on the same sheet of paper draw a curve which shows at every instant the **velocity** of the train? The slope of this new curve would evidently be the acceleration. I am glad to think that nobody has yet given a name to the rate of change of the acceleration.

The symbols in use are

s and t for space and time;

velocity v or $\frac{ds}{dt}$, or Newton's symbol $\dot{s}$;

acceleration $\frac{dv}{dt}$ or $\frac{d^2s}{dt^2}$, or Newton's $\ddot{s}$.

Rate of change of acceleration would be $\frac{d^3s}{dt^3}$.

Note that $\frac{d^2s}{dt^2}$ is one symbol, it has nothing whatever to do with such an algebraic expression as $\frac{d^2 \times s}{d \times t^2}$. The symbol is supposed merely to indicate that we have differentiated s twice with regard to the time.

I have stated that the slope of a curve may be found by drawing a tangent to the curve, and hence it is easy to find the acceleration from the velocity curve.

19. Another way, better than by drawing tangents, is illustrated in this Table:

t seconds	s feet	v feet per second or $\delta s/\delta t$	acceleration in feet per second per second or $\delta v/\delta t$
·06	·0880		
		14·74	
·07	·2354		– 125
		13·49	
·08	·3703		– 127
		12·22	
·09	·4925		– 127
		10·95	
·10	·6020		– 129
		9·66	
·11	·6986		– 131
		8·35	
·12	·7821		– 131
		7·04	
·13	·8525		

In a new mechanism it was necessary for a certain purpose to know in every position of a point A what its acceleration was, and to do this I usually find its velocity first. A skeleton drawing was made and the positions of A marked at the intervals of time t from a time taken as 0. In the table I give at each instant the distance of A from a fixed point of measurement, and I call it s. If I gave the table for all the positions of A till it gets back again to its first position, it would be more instructive, but any student can make out such a table for himself for some particular mechanism. Thus for example, let s be the distance of a piston from the end of its stroke. Of course the all-accomplished mathematical engineer will scorn to take the trouble. He knows a graphical rule for doing this in the case of the piston of a steam engine. Yes, but does he know such a rule for every

possible mechanism? Would it be worth while to seek for such a graphical rule for every possible mechanism? Here is the straightforward Engineers' common-sense way of finding the acceleration at any point of any mechanism, and although it has not yet been tried except by myself and my pupils, I venture to think that it will commend itself to practical men. For beginners it is invaluable.

Now the mass of the body whose centre moves like the point A, being m (**the weight of the body in pounds at London, divided by 32·2**)*, multiply the acceleration in feet per second per second which you find, by m, and you have the force which is acting on the body *increasing* the velocity. The force will be in pounds.

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* I have given elsewhere my reasons for using in books intended for engineers, the units of force employed by all practical engineers. I have used this system (which is, after all, a so-called *absolute* system, just as much as the C. G. S. system or the **Poundal** system of many text books) for twenty years, with students, and this is why their knowledge of mechanics is not a mere book knowledge, something apart from their practical work, but fitting their practical work as a hand does a glove. One might as well talk Choctaw in the shops as speak about what some people call the English system, as if a system can be English which speaks of so many poundals of force and so many foot-poundals of work. And yet these same philosophers are astonished that practical engineers should have a contempt for book *theory*. I venture to say that there is not one practical engineer in this country, who thinks in Poundals, although all the books have used these units for 30 years.

In Practical Dynamics one second is the unit of time, one foot is the unit of space, one pound (what is called the weight of 1 lb. in London) is the unit of force. To satisfy the College men who teach Engineers, I would say that "The unit of Mass is that mass on which the force of 1 lb. produces an acceleration of 1 ft. per sec. per sec."

We have no name for unit of mass, the Engineer never has to speak of the inertia of a body by itself. His instructions are "In all Dynamical calculations, divide the weight of a body in lbs. by 32·2 and you have its mass in Engineer's units—in those units which will give all your answers in the units in which an Engineer talks." If you do not use this system every answer you get out will need to be divided or multiplied by something before it is the language of the practical man.

Force in pounds is the *space*-rate at which work in foot-pounds is done, it is also the *time*-rate at which momentum is produced or destroyed.

Example 1. **A Hammer head** of $2\frac{1}{2}$ lbs. moving with a velocity of 40 ft. per sec. is stopped in ·001 sec. What is the average force of the blow? Here the mass being $2\frac{1}{2} \div 32{\cdot}2$, or ·0776, the momentum (momentum is mass × velocity) destroyed is 3·104. Now force is momentum per sec. and hence the average force is $3{\cdot}104 \div {\cdot}001$ or 3104 lbs.

Example 2. Water in a jet flows with the linear velocity of 20 ft. per sec.

20. We considered the case of falling bodies in which space and time are connected by the law $s=\frac{1}{2}gt^2$, where g the acceleration due to gravity is 32·2 feet per second per second at London. But many other pairs of things are connected by similar laws and I will indicate them generally by

$$\mathbf{y = ax^2}.$$

Let a particular value of a be taken, say $a=\frac{1}{30}$. Now take $x=0$, $x=1$, $x=2$, $x=3$, &c. and in every case calculate y.

Plot the corresponding points on squared paper. They lie on a parabolic curve. At any point on the curve, say where $x=3$, find the slope of the curve $\left(\text{I call it } \frac{dy}{dx}\right)$, do the same at $x=4$, $x=2$, &c. Draw a new curve, now, with the same values of x but with $\frac{dy}{dx}$ as the ordinate. This curve shows at a glance (by the height of its ordinate) what is the slope of the first curve. If you ink these curves, let the y curve be black and the $\frac{dy}{dx}$ curve be red. Notice that the slope or $\frac{dy}{dx}$ at any point, is $2a$ multiplied by the x of the point.

We can investigate this algebraically. As before, for any value of x calculate y. Now take a greater value of x which I shall call $x+\delta x$ and calculate the new y, calling it $y+\delta y$. We have then

$$y+\delta y = a\,(x+\delta x)^2$$
$$= a\,\{x^2 + 2x\,.\,\delta x + (\delta x)^2\}.$$

Subtracting; $\qquad \delta y = a\,\{2x\,.\,\delta x + (\delta x)^2\}.$

Divide by δx, $\qquad \dfrac{\delta y}{\delta x} = 2ax + a\,.\,\delta x.$

(relatively to the vessel from which it flows), the jet being 0·1 sq. ft. in cross section; what force acts upon the vessel?

Here we have $20\times{\cdot}1$ cu. ft. or $20\times{\cdot}1\times 62{\cdot}3$ lbs. of water per sec. or a mass per second in Engineers' units of $20\times{\cdot}1\times 62{\cdot}3\div 32{\cdot}2$. This mass is 3·87, its momentum is 77·4, and as this momentum is lost by the vessel every second, it is the **force acting on the vessel.**

A student who thinks for himself will see that this force is the same whether a vessel is or is not in motion itself.

Imagine δx to get smaller and smaller without limit and use the symbol $\frac{dy}{dx}$ for the limiting value of $\frac{\delta y}{\delta x}$, and we have $\frac{dy}{dx} = 2ax$, a fact which is known to us already from our squared paper*.

21. Note that when we repeat the process of differentiation we state the result as $\frac{d^2y}{dx^2}$ and the answer is $2a$. You must become familiar with these symbols. If y is a function of x, $\frac{dy}{dx}$ is the rate of change of y with regard to x; $\frac{d^2y}{dx^2}$ is the rate of change of $\frac{dy}{dx}$ with regard to x.

Or, shortly; $\frac{d^2y}{dx^2}$ is the **differential coefficient** of $\frac{dy}{dx}$ with regard to x; $\frac{dy}{dx}$ is the differential coefficient of y with regard to x.

Or, again; **integrate** $\frac{d^2y}{dx^2}$ and our answer is $\frac{dy}{dx}$; integrate $\frac{dy}{dx}$ and our answer is y.

You will, I hope, get quite familiar with these symbols and ideas. I am only afraid that when we use other letters than x's and y's you may lose your familiarity.

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* *Symbolically.* Let $y = f(x)$...(1), where $f(x)$ stands for any expression containing x. Take any value of x and calculate y. Now take a slightly greater value of x say $x + \delta x$ and calculate the new y; call it $y + \delta y$

then $$y + \delta y = f(x + \delta x) \quad \text{.............................(2).}$$

Subtract (1) from (2) and divide by δx.

$$\frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x} \quad \text{.............................(3).}$$

What we mean by $\frac{dy}{dx}$ is **the limiting value of** $\frac{f(x + \delta x) - f(x)}{\delta x}$ as δx is made smaller and smaller without limit. This is the exact definition of $\frac{dy}{dx}$. It is quite easy to remember and to write, and the most ignorant person may get full marks for an answer at an examination. It is easy to see that the differential coefficient of $af(x)$ is a times the differential coefficient of $f(x)$ and also that the differential coefficient of $f(x) + F(x)$ is the sum of the two differential coefficients.

The differential coefficient of

$$y = a + bx + cx^2,$$

where a, b and c are constants, is

$$\frac{dy}{dx} = 0 + b + 2cx.$$

The integral of $0 + b + kx$ with regard to x is $A + bx + \frac{1}{2}kx^2$, where A is any constant whatsoever.

Similarly, the integral of $b + kz$ with regard to z is

$$A + bz + \tfrac{1}{2}kz^2.$$

The integral of $b + kv$ with regard to v, is $A + bv + \frac{1}{2}kv^2$.

It is quite easy to work out as an exercise that if $y = ax^3$, then $\frac{dy}{dx} = 3ax^2$, and again that if $y = ax^4$, then $\frac{dy}{dx} = 4ax^3$. All these are examples of the fact that if $y = ax^n$, then $\frac{dy}{dx} = nax^{n-1}$.

In working out any of these examples we take it that $\frac{\delta y}{\delta x}$ becomes $\frac{dy}{dx}$ or that $\delta y = \delta x \times \frac{dy}{dx}$ more and more nearly as δx gets smaller and smaller without limit.

This is sometimes written $y + \delta y = y + \delta x \cdot \frac{dy}{dx}$, or

$$f(x + \delta x) = f(x) + \delta x \frac{df(x)}{dx} \quad \text{.................... (1).}$$

22. Uniformly accelerated motion.

If acceleration, $$\frac{d^2s}{dt^2} = a \quad \text{........................... (1).}$$

Integrate and we have $\frac{ds}{dt} = b + at =$ velocity v. Observe that we have added a constant b, because if we differentiate a constant the answer is 0. There must be some information given us which will enable us to find what the value of the constant b is. Let the information be $v = v_0$, when $t = 0$. Then b is evidently v_0.

So that $$\text{velocity } v = \frac{ds}{dt} = v_0 + at \quad \text{.................. (2).}$$

Again integrate and $s = c + v_0 t + \frac{1}{2}at^2$. Again you will notice that we add an unknown constant, when we integrate. Some information must be given us to find the value of the constant c. Thus if $s = s_0$ when $t = 0$, this s_0 is the value of c, and so we have the most complete statement of the motion

$$s = s_0 + v_0 t + \tfrac{1}{2}at^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

If (3) is differentiated, we obtain (2) and if (2) is differentiated we obtain (1).

23. We see here, then, that **as soon as the student is able to differentiate and integrate** he can work the following kinds of problem.

I. If s is given as any function of the time, differentiate and the velocity at any instant is found; differentiate again and the acceleration is found.

II. If the acceleration is given as some function of the time, integrate and we find the velocity; integrate again and we find the space passed through.

Observe that s instead of being mere distance may be the angle described, the motion being angular or rotational. Better then call it θ. Then $\dot{\theta}$ or $\frac{d\theta}{dt}$ is the angular velocity and $\ddot{\theta}$ or $\frac{d^2\theta}{dt^2}$ is the angular acceleration.

24. Exercises on Motion with constant Acceleration.

1. The acceleration due to gravity is *downwards* and is usually called g, g being 32·2 feet per second per second at London. If a body at time 0 is thrown vertically upwards with a velocity of V_0 feet per second; where is it at the end of t seconds? If s is measured *upwards*, the acceleration is $-g$ and $s = V_0 t - \frac{1}{2}gt^2$. (We assume that there is no resistance of the atmosphere and that the true acceleration is g downwards and constant.)

Observe that $v = V_0 - gt$ and that $v = 0$ when $V_0 - gt = 0$ or $t = \frac{V_0}{g}$. When this is the case find s. This gives the highest point and the time taken to reach it.

When is $s = 0$ again? What is the velocity then?

2. The body of Exercise 1 has been given, in addition to its vertical velocity, a horizontal velocity u_0 which keeps constant. If x is the horizontal distance of it away from the origin at the time t, $\frac{d^2x}{dt^2}=0$ and $\frac{dx}{dt}=u_0$, $x=u_0t$. If we call s by the new name y, we have at any time t,

$$y = V_0t - \tfrac{1}{2}gt^2,$$

$$x = u_0t,$$

and if we eliminate t, we find $y=\frac{V_0}{u_0}x-\frac{1}{2}g\frac{x^2}{u_0^2}$ which is a Parabola.

3. If the body had been given a velocity V in the direction α above the horizontal, we may use $V \sin\alpha$ for V_0 and $V \cos\alpha$ for u_0 in the above expressions, and from them we can make all sorts of useful calculations concerning projectiles.

Plot the curve when $V=1000$ feet per second and $\alpha=45°$.

Again plot with same V when $\alpha=60°$, and again when $\alpha=30°$.

25. **Kinetic Energy.** A small body of mass m is at $s=0$ when $t=0$ and its velocity is v_0, and a force F acts upon it causing an acceleration F/m. As in the last case at any future time

$$v=v_0+\frac{F}{m}t\ldots(1), \text{ and } s=0+v_0t+\tfrac{1}{2}\frac{F}{m}t^2\ldots\ldots(2),$$

(2) may be written $s=\frac{1}{2}t\left(2v_0+\frac{F}{m}t\right)$ and it is easy to see from (1) that this is $s=\frac{1}{2}t(v_0+v)$, and that the average velocity in any interval is half the sum of the velocities at the beginning and end of the interval. Now the work done by the force F in the distance s is Fs. Calculating F from (1), $F=(v-v_0)\frac{m}{t}$ and multiplying upon s we find that the work is $\frac{1}{2}m(v^2-v_0^2)$ which expresses the work stored up in a moving body in terms of its velocity. In fact the work done causes $\frac{1}{2}mv_0^2$ to increase to $\frac{1}{2}mv^2$ and this is the reason why $\frac{1}{2}mv^2$ is called the kinetic energy of a body.

Otherwise. Let a small body of mass m and velocity v pass through the very small space δs in the time δt gaining velocity δv and let a force F be acting upon it. Now $F = m \times$ acceleration or $F = m\frac{\delta v}{\delta t}$ and $\delta s = v \,.\, \delta t$ so that

$$F \,.\, \delta s = mv\delta t \frac{\delta v}{\delta t} = m \,.\, v\delta v = \delta E,$$

if δE stands for the increase in the kinetic energy of the body $\frac{\delta E}{\delta v} = m \,.\, v.$ But our equations are only entirely true when δs, δt, &c., are made smaller and smaller without limit: Hence as $\frac{dE}{dv} = mv$, or in words, "the differential coefficient of E with regard to v is mv," if we integrate with regard to v, $E = \frac{1}{2}mv^2 + c$ where c is some constant. Let $E = 0$ when $v = 0$ so that $c = 0$ and we have $E = \frac{1}{2}mv^2$.

Practise differentiation and integration using other letters than x and y. In this case $\frac{dE}{dv}$ stands for our old $\frac{dy}{dx}$. If we had had $\frac{dy}{dx} = mx$ it might have been seen more easily that $y = \frac{1}{2}mx^2 + c$, but you must escape from the swaddling bands of x and y.

26. Exercise. If x is **the elongation of a spring** when a force F is applied and if $x = \frac{F}{a}$, a representing the stiffness of the spring; $F \,.\, \delta x$ is the work done in elongating the spring through the small distance δx. If F is gradually increased from 0 to F and the elongation from 0 to x, what strain energy is stored in the spring?

The gain of energy from x to $x + \delta x$ is $\delta E = F \,.\, \delta x$, or rather $\frac{dE}{dx} = F = ax$, hence $E = \frac{1}{2}ax^2 + c$. Now if $E = 0$ when $x = 0$, we see that $c = 0$, so that the energy E stored is $E = \frac{1}{2}ax^2 = \frac{1}{2}Fx \ldots(1)$.

It is worth noting that when a mass M is vibrating at the end of a spiral spring; when it is at the distance x from its

position of equilibrium, the potential energy is $\frac{1}{2}ax^2$ and the kinetic energy is $\frac{1}{2}Mv^2$ or the **total energy** is $\frac{1}{2}Mv^2 + \frac{1}{2}ax^2$....

Note that when a force F is required to produce an elongation or compression x in a rod, or a deflexion x in a beam, and if $F = ax$ where a is some constant, the energy stored up as strain energy or potential energy is $\frac{1}{2}ax^2$ or $\frac{1}{2}Fx$.

Also if a **Torque** T is required to produce a turning through the angle θ in a shaft or spring or other structure, and if $T = a\theta$, the energy stored up as strain energy or potential energy is $\frac{1}{2}a\theta^2$ or $\frac{1}{2}T\theta$. If T is in pound-feet and θ is in radians, the answer is in foot-pounds.

Work done = Force × distance, or Torque × angle.

27. If the student knows anything about electricity let him translate into ordinary language the improved Ohm's law

$$\mathbf{V = RC + L \,.\, dC/dt} \quad\text{.................}(1).$$

Observe that if R (Ohms) and L (Henries) remain constant, if C and $\frac{dC}{dt}$ are known to us, we know V, and if the law of V, a changing voltage, is known you may see that there must surely be some means of finding C the changing current. Think of L as the back electromotive force in volts when the current increases at the rate of 1 ampere per second.

If the current in the primary of a **transformer,** and therefore the induction in the iron, did not alter, there would be no electro-motive force in the secondary. In fact the E.M.F. in the secondary is, at any instant, the number of turns of the secondary multiplied by the rate at which the induction changes per second. Rate of increase of I per second is what we now call the differential coefficient of I with regard to time. Although L is constant only when there is no iron or else because the induction is small, the correct formula being $V = RC + N\frac{dI}{dt}$ ......(2), it is found that, practically, (1) with L constant is of nearly universal application. See Art. 183.

28. If $\mathbf{y = ax^n}$ and you wish to find $\frac{dy}{dx}$, I am afraid that

I must assume that you know the **Binomial Theorem** which is :—

$$(x+b)^n = x^n + nbx^{n-1} + \frac{n(n-1)}{2}b^2x^{n-2} + \frac{n(n-1)(n-2)}{6}b^3x^{n-3} + \&c.$$

It is easy to show by multiplication that the Binomial Theorem is true when $n=2$ or 3 or 4 or 5, but when $n=\frac{1}{2}$ or $\frac{1}{3}$ or any other fraction, and again, when n is negative, you had better perhaps have faith in my assertion that the Binomial Theorem can be proved.

It is however well that you should see what it means by working out a few examples. Illustrate it with $n=2$, then $n=3$, $n=4$, &c., and verify by multiplication. Again try $n=-1$, and if you want to see whether your series is correct, just recollect that $(x+b)^{-1}$ is $\frac{1}{x+b}$ and divide 1 by $x+b$ in the regular way by long division.

Let us do with our new function of x as we did with ax^2.

Here $y=ax^n$, $y+\delta y = a(x+\delta x)^n = a\{x^n + n\,.\,\delta x\,.\,x^{n-1} + \frac{n(n-1)}{2}(\delta x)^2 x^{n-2} + \text{terms involving higher powers of } \delta x\}$.

Now subtract and divide by δx and you will find

$$\frac{\delta y}{\delta x} = a\left\{n\,.\,x^{n-1} + \frac{n(n-1)}{2}(\delta x)\,x^{n-2} + \&c.\right\}$$

We see now that as δx is made smaller and smaller, in the limit we have only the first term left, all the others having in them δx or $(\delta x)^2$ or higher powers of δx, and they must all disappear in the limit, and hence,

$$\mathbf{\frac{dy}{dx} = nax^{n-1}}.$$

(See Notes p. 159.)

Thus the differential coefficient of x^6 is $6x^5$, of $x^{2\frac{1}{2}}$ it is $2\frac{1}{2}x^{1\frac{1}{2}}$, and of $x^{-\frac{3}{2}}$ it is $-\frac{3}{2}x^{-\frac{5}{2}}$.

When we find the value of the differential coefficient of any given function we are said to *differentiate* it. When given $\frac{dy}{dx}$ to find y we are said to *integrate*. The origin of

the words *differential* and *integral* need not be considered. They are now technical terms.

Differentiate ax^n and we find nax^{n-1}.

Integrate nax^{n-1} and we find $ax^n + c$. We always add a constant when we integrate.

Sometimes we write these, $\frac{d}{dx}(ax^n) = nax^{n-1}$ and

$$\int nax^{n-1} \,.\, dx = ax^n.$$

Observe that we write $\int$ *before* and dx *after* a function when we wish to say that it is to be integrated with regard to x. Both the symbols are needed. At present you ought not to trouble your head as to why these particular sorts of symbol are used*.

You will find presently that it is not difficult to learn how to differentiate any known mathematical function. You will learn the process easily; but *integration* is a process of guessing, and however much practice we may have, experience only guides us in a process of guessing. To some extent one may say that differentiation is like multiplication or raising a number to the 5th power. Integration is like division, or extracting the 5th root. Happily for the engineer he only needs a very few integrals and these are

* When a great number of things have to be added together in an engineer's office—as when a clerk calculates the weight of each little bit of a casting and adds them all up, if the letter w indicates generally any of the little weights, we often use the symbol Σw to mean the sum of them all. When we indicate the sum of an infinite number of little quantities we replace the Greek letter s or Σ by the long English s or $\int$. It will be seen presently that Integration may be regarded as finding a sum of this kind. Thus if y is the ordinate of a curve; a strip of area is $y \,.\, \delta x$ and $\int y \,.\, dx$ means the sum of all such strips, or the whole area. Again, if δm stands for a small portion of the mass of a body and r is its distance from an axis, then $r^2 \,.\, \delta m$ is called the moment of inertia of δm about the axis, and $\Sigma r^2 \,.\, \delta m$ or $\int r^2 \,.\, dm$ indicates the moment of inertia of the whole body about the axis. Or if δV is a small element of the volume of a body and m is its mass per unit volume, then $\int r^2 m \,.\, dV$ is the body's moment of inertia.

well known. As for the rest, he can keep a good long list of them ready to refer to, but he had better practise working them out for himself.

Now one is not often asked to integrate nax^{n-1}. It is too nicely arranged for one beforehand. One is usually asked to integrate bx^m...(1). I know that the answer is $\frac{bx^{m+1}}{m+1}$...(2). How do I prove this? By differentiating (2) I obtain (1), therefore I know that (2) is the integral of (1). Only I ought to add a constant in (2), any constant whatever, an arbitrary constant as it is called, because the differential coefficient of a constant is 0. Students ought to work out several examples, integrating, say, x^7, bx^4, $bx^{\frac{1}{2}}$, $ax^{-\frac{1}{2}}$, $cx^{\frac{3}{2}}$, $ax^{\frac{1}{3}}$. When one has a list of differential coefficients it is not wise to use them in the reversed way as if it were a list of integrals, for things are seldom given so nicely arranged.

For instance $\int 4x^3 \, . \, dx = x^4$. But one seldom is asked to integrate $4x^3$, more likely it will be $3x^3$ or $5x^3$, that is given.

We now have a number of interesting results, but this last one includes the others. Thus if

$$y = x^3 \text{ or } y = x^2 \text{ or } y = x^1 \text{ or } y = x^0 \text{*},$$

we only have examples of $y = x^n$, and it is good for the student to work them out as examples. Thus

$$\frac{dy}{dx} = nx^{n-1}.$$

If $n = 1$ this becomes $1x^0$ or 1. If $n = 0$ it becomes $0x^{-1}$ or 0. But we hardly need a new way of seeing that if y is a constant, its differential coefficient is 0. We know that if

$$y = a + bx + cx^2 + ex^3 + \&c. + gx^n.$$

Then $$\frac{dy}{dx} = 0 + b + 2cx + 3ex^2 + \&c. + ngx^{n-1},$$

with this knowledge we have the means of working quite

* I suppose a student to know that anything to the power 0 is unity. It is instructive to actually calculate by logarithms a high *root* of any number to see how close to 1 the answer comes. A high root means a small power, the higher the root the more nearly does the power approach 0.

half the problems supposed to be difficult, that come before the engineer.

The two important things to remember now, are: If $y=ax^n$, then $\frac{dy}{dx}=nax^{n-1}$; and if $\frac{dy}{dx}=bx^m$, then

$$y=\frac{b}{m+1}x^{m+1}+c,$$

where c is some constant, or

$$\int bx^m dx=\frac{b}{m+1}x^{m+1}+c.$$

I must ask students to try to discover for themselves **illustrations** of the fact that if $y=x^n$, then $\frac{dy}{dx}=nx^{n-1}$. I do not give here such illustrations as happened to suit myself; they suited me because they were my own discovery. I would suggest this, however:

Take $y=x^5$. Let $x=1{\cdot}02$, calculate y by logarithms. Now let $x=1{\cdot}03$ and calculate y. Now divide the increment of y by $\cdot 01$, which is the increment of x.

Let the second x be $1{\cdot}021$, and repeat the process.

Let the second x be $1{\cdot}0201$, and repeat the process.

It will be found that $\frac{\delta y}{\delta x}$ is approaching the true value of $\frac{dy}{dx}$ which is $5\,(1{\cdot}02)^4$.

Do this again when $y=x^{0{\cdot}7}$ for example. A student need not think that he is likely to waste time if he works for weeks in manufacturing numerical and graphical illustrations for himself. Get really familiar with the simple idea that if $y=x^n$ then $\frac{dy}{dx}=nx^{n-1}$;

that $$\int ax^s\,.\,dx=\frac{a}{s+1}x^{s+1}+\text{constant};$$

that $$\int av^s\,.\,dv=\frac{a}{s+1}v^{s+1}+\text{constant}.$$

Practise this with $s=\cdot 7$ or $\cdot 8$ or $1{\cdot}1$ or $-\,5$ or $-\,\cdot 8$, and use other letters than x or v.

29. Exercises. Find the following Integrals. The constants are not added.

$\int x^2 \, . \, dx$. Answer, $\frac{1}{3}x^3$. $\int v^2 \, . \, dv$. Answer, $\frac{1}{3}v^3$.

$\int v^{-s} \, . \, dv$. The answer is $\frac{1}{1-s} v^{1-s}$.

$\int \sqrt[3]{v^2} \, . \, dv$ or $\int v^{\frac{2}{3}} \, . \, dv$. Answer, $\frac{3}{5}v^{\frac{5}{3}}$.

$\int t^{-\frac{1}{2}} \, . \, dt$. Answer, $2t^{\frac{1}{2}}$.

$\int \frac{1}{x} dx$ or $\int x^{-1} \, . \, dx$. Here the rule fails to help us for we get $\frac{x^0}{0}$ which is ∞, and as we can always subtract an infinite constant our answer is really indeterminate. In our work for some time to come we need this integral in only one case. Later, we shall prove that

$$\int \frac{1}{x} dx = \log x, \text{ and } \int \frac{1}{x+a} dx = \log (x+a),$$

and if $y = \log x$, $\frac{dy}{dx} = \frac{1}{x}$ and $\int \frac{1}{v} dv = \log v$.

If $p = av^3$, then $\frac{dp}{dv} = 3av^2$.

If $v = mt^{-\frac{1}{2}}$, then $\frac{dv}{dt} = -\frac{1}{2}mt^{-\frac{3}{2}}$.

30. If $pv = Rt$, where R is a constant. Work the following exercises. Find $\frac{dp}{dt}$, if v is constant. Answer, $\frac{R}{v}$.

Find $\frac{dv}{dt}$, if p is constant. Answer, $\frac{R}{p}$.

The student knows already that the three variables p, v and t are the pressure volume and absolute temperature of a gas. It is too long to write "$\frac{dp}{dt}$ when v is constant." We

shall use for this the symbol $\left(\frac{dp}{dt}\right)$, the brackets indicating that the variable not there mentioned, is constant.

Find $\left(\frac{dp}{dv}\right)$. Answer, As $p = Rt \,.\, v^{-1}$ we have $\left(\frac{dp}{dv}\right) = -Rtv^{-2}$, and this simplifies to $-pv^{-1}$.

Find $\left(\frac{dv}{dp}\right)$. Answer, As $v = Rt \,.\, p^{-1}$ we have $\left(\frac{dv}{dp}\right) = -Rtp^{-2}$, and this simplifies to $-vp^{-1}$.

Find $\left(\frac{dt}{dp}\right)$. Answer, As $t = \frac{v}{R} \,.\, p$ we have $\left(\frac{dt}{dp}\right) = \frac{v}{R}$.

Find the continued product of the second, fifth, and third of the above answers and meditate upon the fact that

$$\left(\frac{dv}{dt}\right)\left(\frac{dt}{dp}\right)\left(\frac{dp}{dv}\right) = -1.$$

Generally we may say that if u is a function of two variables x and y, or as we say

$$u = f(x, y);$$

then we shall use the symbol $\left(\frac{du}{dx}\right)$ to mean the differential coefficient of u with regard to x when **y is considered to be constant.**

These are said to be **partial differential coefficients.**

31. Here is an excellent exercise for students:—

Write out any function of x and y; call it u.

Find $\left(\frac{du}{dx}\right)$. Now differentiate this with regard to y, assuming that x is constant. The symbol for the result is $\frac{d^2u}{dy \,.\, dx}$.

It will *always* be found that one gets the same answer if one differentiates in the other order, that is

$$\frac{\mathbf{d^2u}}{\mathbf{dy \,.\, dx}} = \frac{\mathbf{d^2u}}{\mathbf{dx \,.\, dy}} \quad \text{.....................(3).}$$

Thus try $u = x^3 + y^3 + ax^2y + bxy^2$,

$$\left(\frac{du}{dx}\right) = 3x^2 + 0 + 2axy + by^2,$$

$$\frac{d^2u}{dy \,.\, dx} = 0 + 0 + 2ax + 2by.$$

Again, $$\left(\frac{du}{dy}\right) = 0 + 3y^2 + ax^2 + 2bxy,$$

and $$\frac{d^2u}{dx \,.\, dy} = 0 + 0 + 2ax + 2by,$$

which is the same as before.

A student ought not to get tired of doing this. Use other letters than x and y, and work many examples. The fact stated in (3) is of enormous importance in Thermodynamics and other applications of Mathematics to engineering. A proof of it will be given later. The student ought here to get familiar with the importance of what will then be proved.

32. One other thing may be mentioned. Suppose we have given us that u is a function of x and y, and that

$$\left(\frac{du}{dx}\right. = ax^3 + by^3 + cx^2y + gxy^2.$$

Then the integral of this is

$$u = \tfrac{1}{4}ax^4 + by^3x + \tfrac{1}{3}cx^3y + \tfrac{1}{2}gx^2y^2 + f(y),$$

where $f(y)$ is some arbitrary function of y. This is added because we always add a constant in integration, and as y is regarded as a constant in finding $\left(\frac{du}{dx}\right)$ we add $f(y)$, which may contain the constant y in all sorts of forms multiplied by constants.

33. To illustrate the fact, still unproved, that if $y = \log x$, then $\frac{dy}{dx} = \frac{1}{x}$. A student ought to take such values of x as 3, 3·001, 3·002, 3·003 &c., find y in every case, divide increments of y by the corresponding increments of x, and see if our rule holds good.

Note that when a mathematician writes $\log x$ he always means the Napierian logarithm of x.

34. Example of $\int \frac{dt}{t} = \log t + \text{constant}$.

It is proved in Thermodynamics that if in a heat engine the working stuff receives heat H at temperature t, and if t_0 is the temperature of the refrigerator, then the work done by a perfect heat engine would be

$$H \cdot \frac{t - t_0}{t}, \text{ or } H\left(1 - \frac{t_0}{t}\right).$$

If one pound of water at t_0 is heated to t_1, and we assume that the heat received per degree is constant, being 1400 foot-lbs.; what is the work which a perfect heat engine would give out in equivalence for the total heat? Heat energy is to be expressed in foot-pounds.

To raise the temperature from t to $t + \delta t$ the heat is $1400\delta t$ in foot-lb. This stands for H in the above expression. Hence, for this heat we have the equivalent work $\delta W = 1400\delta t\left(1 - \frac{t_0}{t}\right)$, or, rather,

$$\frac{dW}{dt} = 1400 - 1400\frac{t_0}{t}.$$

Hence $W = 1400t - 1400t_0 \log t + \text{constant}$.

Now $W = 0$ when $t = t_0$,

$$0 = 1400t_0 - 1400t_0 \log t_0 + \text{constant},$$

therefore the constant is known. Using this value we find equivalent work for the heat given from t_0 to $t_1 = 1400(t_1 - t_0) - 1400t_0 \log \frac{t_1}{t_0}$.

If now the pound of water at t_1 receives the heat L_1 foot-lb. (usually called Latent Heat) and is all converted into steam at the constant temperature t_1, the work which is thermodynamically equivalent to this is $L_1\left(1 - \frac{t_0}{t_1}\right)$. We see then that the work which a **perfect steam engine** would give out as equivalent to the heat received, in raising the pound of water from t_0 to t_1 and then evaporating it, is

$$1400(t_1 - t_0) - 1400t_0 \log_\epsilon \frac{t_1}{t_0} + L_1\left(1 - \frac{t_0}{t_1}\right).$$

Exercise. What work would a perfect steam engine perform per pound of steam at $t_1 = 439$ (or 102 lb. per sq. inch), or 165° C, if $t_0 = 374$ or 100° C. Here $L_1 = 681{,}456$ foot-pounds.

The work is found to be 107,990 ft.-lb. per lb. of steam. Engineers usually wish to know how many pounds of steam are used per hour per Indicated Horse Power. w lb. per hour, means $\frac{107{,}990}{60} w$ ft.-lb. per minute. Putting this equal to 33000 we find w to be 18·35 lb. of steam per hour per Indicated Horse Power, as the requirement of a perfect steam engine working between the temperatures of 165° C and 100° C.

35. Exercises. It is proved in Thermodynamics when ice and water or water and steam are together at the same temperature, if s_1 is the volume of unit mass of stuff in the higher state and s_0 is the volume of unit mass of stuff in the lower state. Then

$$L = t\,(s_1 - s_0)\,\frac{dp}{dt},$$

where t is the absolute temperature, being $274 + \theta°$ C., L being the latent heat in unit mass in foot-pounds. If we take L as the latent heat of 1 lb. of stuff, and s_1 and s_0 are the volumes in cubic feet of 1 lb. of stuff, the formula is still correct, p being in lb. per sq. foot.

I. In Ice-water, $s_0 = \cdot 01747$, $s_1 = \cdot 01602$ at $t = 274$ (corresponding to 0° C.), p being 2116 lb. per sq. foot, and $L = 79 \times 1400$. Hence $\frac{dp}{dt} = -278100$.

And hence the temperature of melting ice is less as the pressure increases; or pressure lowers the melting point of ice; that is, induces towards melting the ice. Observe the quantitative meaning of $\frac{dp}{dt}$; the melting point lowers at the rate of ·001 of a degree for an increased pressure of 278 lb. per sq. foot or nearly 2 lb. per sq. inch.

II. Water Steam. It seems almost impossible to measure accurately by experiment, s_1 the volume in cubic feet of one pound of steam at any temperature. s_0 for water is known. Calculate $s_1 - s_0$ from the above formula, at a few temperatures having from Regnault's experiments the following table. I think that the figures explain themselves.

$\theta°$ C	t absolute	pressure in lb. per sq. inch	p lb. per sq. foot	$\frac{\delta p}{\delta t}$	assumed $\frac{dp}{dt}$	L in foot-pounds	$s_1 - s_0$
100	374	14·70	2116·4				
				81·5			
105	379	17·53	2524		87·8	740,710	22·26
				94			
110	384	20·80	2994				

It is here assumed that the value of dp/dt for 105° C. is half the sum of 81·5 and 94. The more correct way of proceeding would be to plot a great number of values of $\delta p/\delta t$ on squared paper and get dp/dt for 105° C. more accurately by means of a curve. †

$s_1 - s_0$ for 105° C. $= 740710 \div (379 \times 87{\cdot}8) = 22{\cdot}26$. Now $s_0 = {\cdot}016$ for cold water and it is not worth while making any correction for its warmth. Hence we may take $s_1 = 22{\cdot}28$ which is sufficiently nearly the correct answer for the present purpose.

Example. Find s_1 for 275° F. from the following, L being

t°F.	248°	257°	266°	275°	284°	293°	302°
p	4152	4854	5652	6551	7563	8698	9966

Example. If the formula for steam pressure, $p = a\theta^b$ where a and b are known numbers, and θ is the temperature measured from a certain zero which is known, is found to be a useful but incorrect formula for representing Regnault's experimental results; deduce a formula for the volume s_1 of one pound of steam. We have also the well known formula for latent heat $L = c - et$, where t is the absolute temperature and c and e are known numbers. Hence, as $\frac{dp}{d\theta}$ which is the same as $\frac{dp}{dt}$ is $ba\theta^{b-1}$, $s_1 - s_0 = (c - et) \div tba\theta^{b-1}$.

After subjecting an empirical formula to mathematical operations it is wise to test the accuracy of the result on actual experimental numbers, as the formula represents facts only approximately, and the small and apparently insignificant terms in which it differs from fact, may become greatly magnified in the mathematical operations.

36. Study of Curves. When the equation to a new curve is given, the practical man ought to rely first upon his power of plotting it upon squared paper.

Very often, if we find $\frac{dy}{dx}$ or the slope, everywhere, it gives us a good deal of information.

If we are told that x_1, y_1 is a point on a curve, and we are asked to find the equation to the **tangent** there, we have simply to find the straight line which has the same slope as the curve there and which passes through x_1, y_1. The **normal** is the straight line which passes through x_1, y_1 and whose slope is minus the reciprocal of the slope of the curve there. See Art. 13.

P (fig. 8) is a point in a curve APB at which the tangent PS and the normal PQ are drawn. OX and OY are

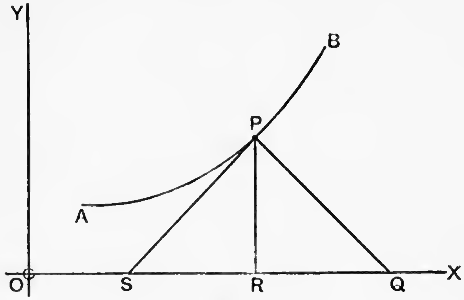

Fig. 8.

the axes. $OR = x$, $RP = y$, $\tan PSR = \dfrac{dy}{dx}$; the distance SR is called the **subtangent**; prove that it is equal to $y \div \dfrac{dy}{dx}$. The distance RQ is called the **subnormal**; it is evidently equal to $y\dfrac{dy}{dx}$. The length of the tangent PS will be found to be $y\sqrt{1+\left(\dfrac{dx}{dy}\right)^2}$, the length of the normal PQ is $y\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$. The Intercept OS is $x - y\dfrac{dx}{dy}$.

Example 1. Find the length of the subtangent and subnormal of the Parabola $\quad y = mx^2$,

$$\frac{dy}{dx} = 2mx.$$

Hence $\quad$ Subtangent $= mx^2 \div 2mx$ or $\frac{1}{2}x$.

$\quad$ Subnormal $= y \times 2mx$ or $2m^2x^3$.

Example 2. Find the length of the subtangent of $y = mx^n$,

$$\frac{dy}{dx} = mnx^{n-1}.$$

$$\text{Subtangent} = mx^n \div mnx^{n-1} = x/n.$$

Example 3. Find of what curve the subnormal is constant in length,

$$y\frac{dy}{dx}=a \text{ or } \frac{dx}{dy}=\frac{1}{a}y.$$

The integral of $\frac{1}{a}y$ with regard to y is $x=\frac{1}{2a}y^2+$ a constant b, and this is the equation to the curve, where b may have any value. It is evidently one of a family of parabolas. (See Art. 9 where x's and y's are merely interchanged.)

Example 4. The point $x=4$, $y=3$ is a point in the parabola $y=\frac{3}{2}x^{\frac{1}{2}}$. Find the equation to the tangent there. The slope is $\frac{dy}{dx}=\frac{1}{2}\times\frac{3}{2}x^{-\frac{1}{2}}$ or, as $x=4$ there, the slope is $\frac{3}{4}\times\frac{1}{2}$ or $\frac{3}{8}$. The tangent is then, $y=m+\frac{3}{8}x$. To find m we have $y=3$ when $x=4$ as this point is in the tangent, or $3=m+\frac{3}{8}\times 4$, so that m is $1\frac{1}{2}$ and the tangent is $y=1\frac{1}{2}+\frac{3}{8}x$.

Example 5. The point $x=32$, $y=3$ is evidently a point in the curve $y=2+\frac{1}{2}x^{\frac{1}{5}}$. Find the equation to the normal there.

The slope of the curve there is $\frac{dy}{dx}=\frac{1}{10}x^{-\frac{4}{5}}=\frac{1}{160}$ and the slope of the normal is minus the reciprocal of this or -160. Hence the normal is $y=m-160x$. But it passes through the point $x=32$, $y=3$ and hence $3=m-160\times 32$.

Hence $m=5123$ and the normal is $y=5123-160x$.

Example 6. At what point in the curve $y=ax^{-n}$ is there the slope b?

$$\frac{dy}{dx}=-nax^{-n-1}.$$

The point is such that its x satisfies $-nax^{-n-1}=b$ or, $x=\left(-\frac{na}{b}\right)^{\frac{1}{n+1}}$. Knowing its x we know its y from the equation to the curve. It is easy to see and well to remember that if x_1, y_1 is a point in a straight line, and if the slope of the line is b, then the equation to the line most quickly written is

$$\frac{y-y_1}{x-x_1}=b.$$

Hence the equation to the tangent to a curve at the point x_1, y_1 on the curve is

$$\frac{y-y_1}{x-x_1} = \text{the } \frac{dy}{dx} \text{ at the point.}$$

And the equation to the normal is

$$\frac{y-y_1}{x-x_1} = -\text{ the } \frac{dx}{dy} \text{ at the point.}$$

Exercise 1. Find the tangent to the curve $x^m y^n = a$ at the point x_1, y_1 on the curve. Answer, $\frac{m}{x_1}x + \frac{n}{y_1}y = m + n$.

Exercise 2. Find the normal to the same curve.

$$\text{Answer, } \frac{n}{y_1}(x-x_1) - \frac{m}{x_1}(y-y_1) = 0.$$

Exercise 3. Find the tangent and normal to the parabola $y^2 = 4ax$ at the point where $x = a$.

Answer, $y = x + a$, $y = 3a - x$.

Exercise 4. Find the tangent to the curve

$$y = a + bx + cx^2 + ex^3$$

at a point on the curve x_1, y_1.

$$\text{Answer, } \frac{y-y_1}{x-x_1} = b + 2cx_1 + 3ex_1^2.$$

37. When y increases to a certain value and then diminishes, this is said to be a **maximum** value of y; when y diminishes to a certain value and then increases, this is said to be a **minimum** value of y. It is evident that for either case $\frac{dy}{dx} = 0$. See Art. 16 and fig. 6.

Example 1. Divide 12 into two parts such that the product is a maximum. The practical man tries and easily finds the answer. He tries in this sort of way. Let x be one part and $12 - x$ the other. He tries $x = 0$, $x = 1$, $x = 2$, &c., in every case finding the product. Thus

x	0	1	2	3	4	5	6	7	8	9
Product	0	11	20	27	32	35	36	35	32	27

It seems as if $x=6$, giving the product 36, were the correct answer. But if we want to be more exact, it is good to get a sheet of squared paper; call the product y and plot the corresponding values of x and y. The student ought to do this himself.

Now it is readily seen that where y has a maximum or a minimum value, in all cases the slope of a curve is 0. Find then the point or points where dy/dx is 0.

Thus if a number a is divided into two parts, one of them x and the other $a-x$, the product is $y=x(a-x)$ or $ax-x^2$, and $\frac{dy}{dx}=a-2x$. Find where this is 0. Evidently where $2x=a$ or $x=\frac{1}{2}a$.

The practical man has no great difficulty in any of his problems in finding whether it is a maximum or a minimum which he has found. In this case, let $a=12$. Then $x=6$ gives a product 36. Now if $x=5{\cdot}999$, the other part is $6{\cdot}001$ and the product is $35{\cdot}999999$, so that $x=6$ gives a greater product than $x=5{\cdot}999$ or $x=6{\cdot}001$, and hence it is a maximum and not a minimum value which we have found. This is the only method that the student will be given of distinguishing a maximum from a minimum at so early a period of his work.

Example 2. Divide a number a into two parts such that the sum of their squares is a minimum. If x is one part, $a-x$ is the other. The question is then, if

$$y=x^2+(a-x)^2, \text{ when is } y \text{ a minimum?}$$

$$y=2x^2+a^2-2ax,$$

$$\frac{dy}{dx}=4x-2a, \text{ and this is 0 when } x=\tfrac{1}{2}a.$$

Example 3. When is the sum of a number and its reciprocal a minimum? Let x be the number and $y=x+\frac{1}{x}$. When is y a minimum?

The differential coefficient of $\frac{1}{x}$ or x^{-1} being $-x^{-2}$, we have $\frac{dy}{dx}=1-\frac{1}{x^2}$, and this is 0 when $x=1$.

The student ought to take numbers and a sheet of squared paper and try. Trying $x = 100, 10, 4$ &c. we have

x	100	10	4	2	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
y	100·01	10·1	4·25	2·5	2	2·5	$3\frac{1}{3}$	$4\frac{1}{4}$

Now let him plot x and y and he will see that y is a minimum when $x = 1$.

Example 4. **The strength of a rectangular beam** of given length, loaded and supported in any particular way, is proportional to the breadth of the section multiplied by the square of the depth. If the diameter a is given of a cylindric tree, what is the strongest beam which may be cut from it? Let x be its breadth. Then if you draw the rectangle inside the circle, you will see that the depth is $\sqrt{a^2 - x^2}$. Hence the strength is a maximum when y is a maximum if

$$y = x(a^2 - x^2),$$

or

$$y = a^2x - x^3,$$

$$\frac{dy}{dx} = a^2 - 3x^2, \text{ and this is 0 when } x = \frac{a}{\sqrt{3}}.$$

In the same way find the *stiffest* beam which may be cut from the tree by making the breadth × the cube of the depth a maximum.

This, however, may wait till the student has read Chap. III.

Example 5. Experiments on the **explosion of mixtures** (at atmospheric pressure) lead to a roughly correct rule

$$p = 83 - 3{\cdot}2x, \dagger$$

where p is the highest pressure produced in the explosion, and x is the volume of air together with products of previous combustions, added to one cubic foot of coal gas before explosion. Taking px as roughly proportional to the work done in a gas engine during explosion and expansion; what value of x will make this a maximum?

That is, when is $83x - 3{\cdot}2x^2$ a maximum? Answer, When $83 - 6{\cdot}4x = 0$, or x is about 13 cubic feet.

I am afraid to make Mr Grover responsible for the above result which I have drawn from his experiments. His most interesting result was, that of the above 13 cubic feet it is very much better that only 9 or 10 should be air than that it should all be air.

Example 6. Prove that $ax - x^2$ is a maximum when

$$x = \tfrac{1}{2}a.$$

Example 7. Prove that $x - x^3$ is a maximum when

$$x = \tfrac{1}{3}\sqrt{3}.$$

Example 8. The **volume** of a circular cylindric **cistern** being given (no cover) when is its surface a minimum?

Let x be the radius and y the length; the volume is

$$\pi x^2 y = a, \text{ say} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The surface is $$\pi x^2 + 2\pi xy\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

When is this a minimum?

From (1), y is $\dfrac{a}{\pi x^2}$; using this in (2) we see that we must make

$$\pi x^2 + \frac{2a}{x} \text{ a minimum},$$

$$2\pi x - \frac{2a}{x^2} = 0 \text{ or } x^3 = \frac{a}{\pi},$$

$$x^3 = \frac{\pi x^2 y}{\pi} \text{ or } x = y.$$

The radius of the base is equal to the height of the cistern.

Example 9. Let the cistern of Ex. 8 be closed top and bottom, find it of minimum surface and given volume.

The surface is $2\pi x^2 + 2\pi xy$, and proceeding as before we find that the diameter of the cistern is equal to its height.

Example 10. If v is the velocity of water in a river and x is the velocity against stream of a steamer *relatively to the water*, and if the **fuel burnt per hour** is $a + bx^3$; find the

velocity x so as to make the consumption of fuel a minimum for a given distance m. The velocity of the ship relatively to the bank of the river is $x - v$, the time of the passage is $\frac{m}{x-v}$, and therefore the fuel burnt during the passage is $\frac{m(a+bx^3)}{x-v}$.

Observe that $a + bx^3$ with proper values given to a and b may represent the total cost per hour of the steamer, including interest and depreciation on the cost of the vessel, besides wages and provisions.

You cannot yet differentiate a quotient, so I will assume $a = 0$, and the question reduces to this: when is $\frac{x^3}{x-v}$ a minimum? Now this is the same question as:—when is $\frac{x-v}{x^3}$ a maximum? or when is $x^{-2} - vx^{-3}$ a maximum? The differential coefficient is $-2x^{-3} + 3vx^{-4}$. Putting this equal to 0 we find $x = \frac{3}{2}v$, or that the speed of the ship relatively to the water is half as great again as that of the current.

Notice here as in all other cases of maximum and minimum that the engineer ought not to be satisfied merely with such an answer. $x = \frac{3}{2}v$ is undoubtedly the best velocity, it makes $x^3/(x - v)$ a minimum. But suppose one runs at less or more speed than this, does it make much difference? Let $v = 6$, the best x is 9,

$$\frac{x^3}{x-6} = 243 \text{ if } x = 9.$$

$$= 250 \text{ if } x = 10.$$

$$= 256 \text{ if } x = 8;$$

and these figures tell us the nature of the extra expense in case the theoretically correct velocity is not adhered to*.

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* Assuming that you know the rule for the differentiation of a quotient —usually learnt at the very beginning of one's work in the Calculus, and without assuming a to be 0 as above, we have

$$(x - v)\, 3bx^2 = a + bx^3,$$

$$2bx^3 - 3bvx^2 = a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Example 11. The sum of the squares of two factors of a is a minimum, find them. If x is one of them, $\frac{a}{x}$ is the other, and $y = x^2 + \frac{a^2}{x^2}$ is to be a minimum, $\frac{dy}{dx} = 2x - \frac{2a^2}{x^3}$, and this is 0 when $x^4 = a^2$ or $x = \sqrt{a}$.

Example 12. To arrange n **voltaic cells** so as to obtain the maximum current through a resistance R. Let the E.M.F. of each cell be e and its internal resistance r. If the cells are arranged as x in series, n/x in parallel, the E.M.F. of the battery is xe, and its internal resistance is $\frac{x^2 r}{n}$. Hence the current $C = xe \div \left(\frac{x^2 r}{n} + R\right)$.

As the student cannot yet differentiate a quotient, we shall say that C is a maximum when its reciprocal is a minimum, so we ask when is $\left(\frac{x^2 r}{n} + R\right) \div xe$ or $\frac{xr}{n} + \frac{R}{x}$ a

---

Given the values of a, b and v the proper value of x can be found by trial. Thus let the cost per day in pounds be $30 + \frac{1}{20}x^3$ so that $a = 30$, $b = \frac{1}{20}$ and let $v = 6$. Find x from (1) which becomes

$$x^3 - 9x^2 - 300 = 0 \quad \text{.............................(2)}.$$

I find that $x = 11{\cdot}3$ is about the best answer.

This is a cubic equation and so has three roots. But the engineer needs only one root, he knows about how much it ought to be and he only wants it approximately. **He solves any equation whatsoever** in the following sort of way.

Let $x^3 - 9x^2 - 300$ be called $f(x)$. The question is, what value of x makes this 0? Try $x = 10$, $f(x)$ turns out to be -200,

x	10	8	12	11	11·3
$f(x)$	−200	−360	+176	−57	−6

whereas we want it to be 0. Now I try $x = 8$, this gives -360 which is further wrong. Now I try 12 and I get 176 so that x evidently lies between 10 and 12. Now I try 11 and find -57. It is now worth while to use squared paper and plot the curve $y = f(x)$ between $x = 10$ and $x = 12$. One can find the true answer to any number of places of decimals by repeating this process. In the present case no great accuracy is wanted and I take $x = 11{\cdot}3$ as the best answer. Note that the old answer obtained by assuming $a = 0$ is only 9. A practical man will find much food for thought in thinking of these two answers. Note that the captain of a river steamer must always be making this sort of calculation although he may not put it down on paper.

minimum? Its differential coefficient is $\frac{r}{n} - \frac{R}{x^2}$ and this is 0 when $R = \frac{x^2 r}{n}$, which is the internal resistance of the battery.

Hence we have the rule: Arrange the battery so that its internal resistance shall be as nearly as possible equal to the external resistance.

Example 13. **Voltaic cell** of E.M.F. $= e$ and internal resistance r; external resistance R. The current is $C = \frac{e}{r+R}$. The power given out is $P = RC^2$. What value of R will make P a maximum? $P = R\frac{e^2}{(r+R)^2}$.

To make this suit such work as we have already done we may say, what value of R will make $\frac{(r+R)^2}{R}$ a minimum, or $\frac{r^2 + 2Rr + R^2}{R}$ or $r^2R^{-1} + 2r + R$ a minimum?

Putting its differential coefficient with regard to R equal to 0 we have $-r^2R^{-2} + 1 = 0$ so that $R = r$, or the external resistance ought to be equal to the internal resistance.

Example 14. What is the volume of the greatest box which may be sent by **Parcels post**? Let x be the length, y and z the breadth and thickness. The P. O. regulation is that the length plus girth must not be greater than 6 feet. That is, we want $v = xyz$ to be a maximum, subject to the condition that $x + 2(y+z) = 6$. It is evident that y and z enter into our expressions in the same way, and hence $y = z$. So that $x + 4y = 6$ and $v = xy^2$ is to be a maximum. Here as $x = 6 - 4y$ we have $v = (6-4y)y^2$ or $6y^2 - 4y^3$ to be a maximum. Putting $\frac{dv}{dy} = 0$ we have $12y - 12y^2 = 0$. Rejecting $y = 0$ for an obvious reason, $y = 1$, and hence our box is 2 feet long, 1 foot broad, 1 foot thick, containing 2 cubic feet.

Find the volume of the greatest cylindric parcel which may be sent by Post. Length being l and diameter d, $l + \pi d = 6$ and $\frac{\pi}{4} ld^2$ is to be a maximum. Answer, $l = 2$ feet, $d = \frac{4}{\pi}$ feet, volume $= 8 \div \pi$ or 2·55 cubic feet.

Example 15. **Ayrton-Perry Spring.** Prof. Ayrton and the present writer noticed that in a spiral spring fastened at one end, subjected to axial force F, the free end tended to rotate. Now it was easy to get the general formula for the elongation and rotation of a spring of given dimensions, and by nothing more than the above principle we found what these dimensions ought to be for the rotation to be great.

Thus for example, the angle of the spiral being α the rotation was proportional to $\sin\alpha\cos\alpha$. It at once followed that α ought to be 45°.

Again, the wire being of elliptic section, x and y being the principal radii of the ellipse, we found that the rotation was proportional to

$$\frac{x^2+y^2}{x^3y^3}-\frac{8}{5xy^3}.$$

To make this a maximum, the section (which is proportional to xy) being given. Let $xy=s$, a constant, then the above expression becomes

$$\frac{y^2}{s^3}-\frac{3}{5}\frac{1}{sy^2}, \text{ and this is to be a maximum.}$$

Here we see that there is no true maximum. The larger we make y or the smaller we make y (for small values of y the rotation is negative but we did not care about the direction of our rotation, that is, whether it was with or against the usual direction of winding up of the coils) the greater is the rotation. This is how we were led to make springs of thin strips of metal wound in spirals of 45°. The amount of rotation obtained for quite small forces and small axial elongations is quite extraordinary. The discovery of these very useful springs was complete as soon as we observed that any spring rotated when an axial force was applied. Students who are interested in the practical application of mathematics ought to refer to the complete calculations in our paper published in the *Proceedings of the Royal Society* of 1884.

Example 16. From a Hypothetical **Indicator Diagram** the indicated work done per cubic foot of steam is

$$w=144p_1(1+\log r)-144rp_3-x,$$

where p_1 and p_3 are the initial and back pressures of the steam; r is the ratio of cut off (that is, cut off is at $\frac{1}{r}$th of the stroke) and x is a loss due to **condensation in the cylinder.** x depends upon r.

1st. If x were 0, what value of r would give most indicated work per cubic foot of steam?

We must make $\frac{dw}{dr}=0$, and we find $\frac{144p_1}{r}-144p_3=0$ or $r=\frac{p_1}{p_3}$. If it is *brake* energy which is to be a maximum per cubic foot of steam, we must add to p_3 a term representing engine friction.

2nd. Mr Willans found by experiment in non-condensing engines that $r=\frac{p_1}{p_3+10}$ gave maximum indicated w. Now if we put in the above $\frac{dw}{dr}=0$ we have $\frac{144p_1}{r}-144p_3-\frac{dx}{dr}=0$.

So that $\frac{dx}{dr}=\frac{144p_1}{p_1}(p_3+10)-144p_3$ or $\frac{dx}{dr}=1440$. So that $x=1440r+\text{constant}$. Hence Mr Willans' practical rule leads us to the notion that the work lacking per cubic foot of steam is a linear function of r.

This is given here merely as a pretty exercise in maxima and minima. As to the practical engineering value of the result, much might be said for and against. It really is as if there was an extra back pressure of 10 lb. per sq. inch which represented the effect of condensation.

Mr Willans found experimentally in a non-condensing engine that the missing water per Indicated Horse Power hour is a linear function of r using the same steam in the boiler, but this is not the same as our x. We sometimes assume the ratio of condensed steam to indicated steam to be proportional to log r, but a linear function of r will agree just as well with such experimental results as exist.

Example 17. **The weight of gas which will flow per second through an orifice** from a vessel where it is at

pressure p_0 into another vessel where it is at the pressure p is proportional to $\alpha^{\frac{1}{\gamma}}\sqrt{1-\alpha^{\frac{\gamma-1}{\gamma}}}$; where α is p/p_0 and γ is a known constant, when is this a maximum? That is, when is $\alpha^{\frac{2}{\gamma}}-\alpha^{1+\frac{1}{\gamma}}$ a maximum? See Art. 74, where this example is repeated.

Differentiating with regard to α and equating to 0 we find

$$p=p_0\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}.$$

In the case of air $\gamma=1{\cdot}41$ and we find $p=\cdot527p_0$, that is, there is a maximum quantity leaving a vessel per second when the outside pressure is a little greater than half the inside pressure.

Example 18. Taking the waste going on in **an electric conductor** as consisting of (1) the ohmic loss; the value of C^2r watts, where r is the resistance in ohms of a mile of going and coming conductor and C is the current in amperes; (2) the loss due to interest and depreciation on the cost of the conductor. I have taken the price lists of manufacturers of cables, and contractors' prices for laying cables, and I find that in every case of similar cables, similarly laid, or suspended if overhead, the cost of a mile of conductor is practically proportional to the weight of copper in it, that is, inversely proportional to the resistance, plus a constant. The cost of it per year will depend upon the cost of copper per ton, multiplied by the number taken as representing rate per cent. per annum of interest and depreciation. We can state this loss per year or per second, in money per year or per second and the ohmic loss is in watts. We cannot add them together until we know the money value per year or per second of 1 watt. There are three things then that decide the value of the quantity which we call t^2. I prefer to express the total waste going on in watts rather than in pounds sterling per annum and I find it to be $y=C^2r+\frac{t^2}{r}+b$, where b is some constant. The value of t may be taken as anything from 17 to 40 for the working of exercises, but

students had better take figures of their own for the cost of power, copper and interest*.

For a given current C, when is the y, the total waste, a minimum? that is, what is the most economical conductor for a given current? $\frac{dy}{dr} = C^2 - \frac{t^2}{r^2}$ and this is 0 when $r = \frac{t}{C}$.

Thus if $$t = 40,\quad r = \frac{40}{C}.$$

Now if a is the cross section of the conductor in square inches, $r = \frac{\cdot 04}{a}$ nearly, so that $C = 1000a$, or it is most economical to provide one square inch of copper for every 1000 amperes of current.

When u is a function of **more than one independent**

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* The weight of a mile of copper, a square inches in cross section, is to be figured out. Call it ma tons. If p is the price in pounds sterling of a ton of copper, the price of the cable may be taken as, nearly, $pma +$ some constant. If R is the rate per cent. per annum of interest and depreciation, then the loss per annum due to cost of cable may be expressed in pounds as $\frac{R}{100} pma +$ some constant. If £1 per annum is the value of w watts, (observe that this figure w must be evaluated with care. If the cable is to have a constant current for 24 hours a day, every day, w is easily evaluated), then the cost of the cable leads to a perpetual loss of $\frac{R}{100} wpma +$ some constant. Now taking $a = \frac{\cdot 04}{r}$, we see that our t^2 is $\frac{Rwpm}{2500}$.

Men take the answer to this problem as if it gave them the most economical current for any conductor under all circumstances. But although the above items of cost are most important, perhaps, in long cables, there are **other items of cost** which are not here included. The cost of **nerves and eyesight** and comfort if a light blinks; the cost in the armature of a dynamo of the **valuable space** in which the current has to be carried.

If a man will only write down as a mathematical expression the total cost of any engineering contrivance as a function of the size of one or more variable parts, it is quite easy to find the best size or sizes; but it is not always easy to write down such a function. And yet this is the sort of problem that every clever engineer is always working in his head; increasing something has bad and good effects; what one ought to do is a question in maxima and minima.

Notice also this. Suppose we find a value of x which makes y a maximum; it may be, that quite different values of x from this, give values of y which are not very different from the maximum value. The good practical engineer will attend to matters of this kind and in such cases he will **not insist too strongly** upon the use of a particular value of x.

variable, say x and y. Then $\left(\frac{du}{dx}\right)=0$, y being considered constant during the differentiation, and $\left(\frac{du}{dy}\right)=0$, x being considered constant during the differentiation, give two equations which enable the values of x and y to be found which will make u a maximum or a minimum. Here, however, there is more to be said about whether it is a minimum or a maximum, or a maximum as to x and a minimum as to y, which one has found, and we cannot here enter into it.

Sometimes in the above case although u is a function of x and y, there may be a law connecting x and y, and a little exercise of common sense will enable an engineer to deal with the case. All through our work, that is what is wanted, no mere following of custom; a man's own thought about his own problems will enable him to solve very difficult ones with very little mathematics.

Thus for example, if we do not want to find **the best conductor for a given current of Electricity;** if it is the Power to be delivered at the distant place that is fixed. If the distance is n miles, and the conductors have a resistance of r ohms per mile (go and return), if V_1 is the potential, given, at the Generating end, and C is the current. Then the potential at the receiving end being V; $V_1-V=Cnr$. $CV=P$ is fixed, and the cost per mile is $y=C^2r+\frac{t^2}{r}$...(1), where t^2 is known. When is y a minimum?

Here both C and r may vary, but not independently. $V=V_1-Cnr$ and $P=CV_1-C^2nr$ ... (2). One simple plan is to state y in terms of r alone or of C alone. Thus r from (2) is $r=\frac{CV_1-P}{C^2n}$ ......(3).

Substituting for this in (1), we get

$$y=\frac{CV_1-P}{n}+\frac{t^2C^2n}{CV_1-P} \quad \text{.......................} (4).$$

Here everything is constant except C, so we can find the value of C to make y a minimum, and when we know C we also know r from (3).

At present the student is supposed to be able to differentiate only x^n, so he need not proceed with the problem until he has worked a few exercises in Chap. III.*

* To differentiate (4) is a very easy exercise in Chap. III. and leads to $\frac{dy}{dC}=\frac{V_1}{n}+\frac{(CV_1-P)\,2t^2Cn-t^2C^2nV_1}{(CV_1-P)^2}$, and on putting this equal to 0 we obtain the required value of C. It would not be of much use to proceed further

In my Cantor lectures on Hydraulic Machinery, I wrote out an expression for the total loss in pounds per annum in **Hydraulic transmission of power** by a pipe. I gave it in terms of the maximum pressure, the power sent in, and the diameter d of the pipe. It was easy to choose d to make the total cost a minimum. If however I had chosen p, the pressure at the receiving end as fixed, and the power delivered as fixed, and therefore Q the cubic feet of water per second, and if I had added the cost of laying as proportional to the square of the diameter, I should have had an expression for the total cost like

$$y = a\frac{lQ^3}{d^5} + b\frac{l^2Q^2}{d^3} + cld^2,$$

when the values of a, b and c depend upon the cost of power, the interest on the cost of iron, &c. This is a minimum when its differential coefficient with regard to d is zero or $2cld = 5alQ^3d^{-6} + 3bl^2Q^2d^{-4}$, and d can be obtained by trial. The letters b and c also involved the strength of the material, so that it was possible to say whether wrought iron or cast iron was on the whole the cheaper. But even here a term is neglected, the cost of the Engine and Pumps.

The following example comes in conveniently here, although it is not an example of Maximum or Minimum.

An Electric Conductor gives out continuously a amperes of current in every mile of its length. Let x be the distance of any point in miles from the end of the line remote from the generator, let C be the current there and V the voltage. Let r ohms per mile be the resistance of the conductor (that is, of one mile of going and one mile of coming conductor). The current given out in a distance δx is δC, or rather $\delta x\,\dfrac{dC}{dx}$, and the power is $\delta x\,.\,V\,.\,\dfrac{dC}{dx}$, so that if P is the power per mile (observe the meaning of *per*),

$$P = V\frac{dC}{dx} \quad\text{.............................}(1).$$

Also if V is voltage at x and $V+\delta V$ at $x+\delta x$;—

As the resistance is $r\,.\,\delta x$ the current is $\delta V \div r\,.\,\delta x$, or rather, since these expressions are not correct until δx is supposed smaller and smaller without limit,

$$C = \frac{1}{r}\,\frac{dV}{dx} \quad\text{.............................}(2).$$

unless we had numerical values given us. Thus take $V_1 = 300$ volts, $n = 10$ miles, $P = 20000$ watts, $t^2 = 1600$, find C and then r.

Consult a Paper in the *Journal of the Institution of the Society of Telegraph Engineers*, p. 120, Vol. xv. 1886, if there is any further difficulty.

It has not yet been sufficiently noted that if V_1 and P and r are given, there is a limiting length of line

$$n = V_1{}^2/4rP,$$

and when this is the case P is exactly equal to the ohmic loss in the conductor.

As $\frac{dC}{dx}=a$, $C=ax$ if C is 0 when $x=0$.

Hence if r is constant (2) becomes

$$rax=\frac{dV}{dx} \text{ so that } V=V_0+\tfrac{1}{2}rax^2 \quad\text{.................(3)},$$

V_0 being the voltage at the extremity of the line.

(1) becomes $$P=aV_0+\tfrac{1}{2}ra^2x^2 \quad\text{......................... (4)}.$$

Taking $V_0=200$ volts, $a=25$ amperes per mile, $r=1$ ohm per mile, it is easy to see by a numerical example, how the power dispensed per mile, and the voltage, diminish as we go away from the generator.

x	V	P
0	200	5000
1	212·5	5312
2	250	6250
3	312·5	7812
4	400	10,000

If V_1 is the voltage at the Dynamo and the line is n miles long $V_1=V_0+\frac{1}{2}arn^2$ from (4).

The power per mile at the extremity being $P_0=aV_0$, if we are given V_1 and P_0 to find V_0, we shall find that n cannot be greater than

$$V_1 \div \sqrt{2rP_0},$$

and this gives the limiting length of the line.

If we wish, as in **Electric traction** to get a nearer approach to uniform P, let us try

$$C=ax-bx^c \quad\text{............................ (5)},$$

where a, b, c are constants,

$$\frac{1}{r}\frac{dV}{dx}=ax-bx^c,$$

$$V=V_0+\tfrac{1}{2}rax^2-\frac{rb}{c+1}x^{c+1} \quad\text{................. (6)}.$$

As $P=V\frac{dC}{dx}$, or $V(a-cbx^{c-1})$, we can easily determine the three constants a, b, c so that P shall be the same at any three points of the line. Thus let $r=1$ ohm, $V_0=100$ volts, and let $P=10000$ watts, where $x=0$, $x=1$ mile, $x=1\frac{1}{2}$ miles.

We find by trial that

$$C=100x-14{\cdot}75x^{2{\cdot}115},$$

and from this it is easy to calculate C at any point of the line.

Example 19. A machine costs $ax+by$, its value to me is proportional to xy, find the best values of x and y if the cost is fixed. Here xy is to be a maximum. Let $c=ax+by$, so that $y=\frac{c}{b}-\frac{a}{b}x$, and xy is $\frac{c}{b}x-\frac{a}{b}x^2$. This is a maximum when $\frac{c}{b}=2\frac{a}{b}x$ or $ax=\frac{c}{2}$. Hence $ax=by=c/2$ makes xy a maximum.

Example 20. **The electric time constant** of a cylindric coil of wire is approximately

$$u=mxyz/(ax+by+cz),$$

where x is the mean radius, y is the difference between the internal and external radii, z is the axial length and m, a, b, c are known constants.

The volume of the coil is $2\pi xyz$.

Find the values of x, y, z to make u a maximum if the volume of the coil is fixed. Let then $2\pi . xyz=g$; when is $\frac{a}{yz}+\frac{b}{xz}+\frac{c}{xy}$ a minimum? That is, substituting for z, when is $ax+by+\frac{gc}{2\pi xy}=v$, say, a minimum? As x and y are perfectly independent we put $\left(\frac{dv}{dx}\right)=0$ and $\left(\frac{dv}{dy}\right)=0$,

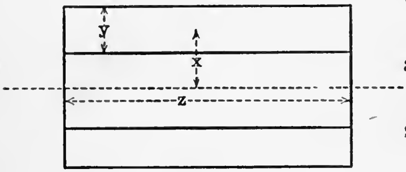

Fig. 9.

$$\text{or} \qquad a+0-\frac{gc}{2\pi yx^2}=0,$$

$$\text{and} \qquad 0+b-\frac{gc}{2\pi xy^2}=0,$$

$$\text{so that} \qquad x^2y=\frac{gc}{a2\pi},$$

$$xy^2=\frac{gc}{b2\pi},$$

and

$$\frac{x}{y}=\frac{b}{a} \text{ or } y=\frac{ax}{b},\ x^2\frac{ax}{b}=\frac{gc}{a2\pi},$$

$$x^3=\frac{bgc}{a^2 2\pi} \text{ or } x=\sqrt[3]{\frac{bgc}{a^2 2\pi}},$$

$$y=\sqrt[3]{\frac{agc}{b^2 2\pi}}, \text{ and } z=\frac{g}{2\pi xy} \text{ or } z=\sqrt[3]{\frac{abg}{c^2 2\pi}}.$$

38. The chain of a suspension bridge supports a load by means of detached rods; the loads are about equal and equally spaced. Suppose a chain to be really continuously loaded, the load being w per unit length *horizontally*. Any very flat uniform chain or telegraph wire is nearly in this condition. What is its shape? Let O be the lowest point. OX is tangential to the chain and horizontal at O. OY is vertical. Let P be any point in the chain, its co-ordinates being x and y. Consider the equilibrium of the portion OP. OP is in equilibrium, under the action of T_0 the horizontal tensile force at O, T the inclined tangential force at P and wx the resultant load upon OP acting vertically. We employ the laws of forces acting upon rigid bodies. A rigid body is a body which is acted on by forces and is no longer altering its shape.

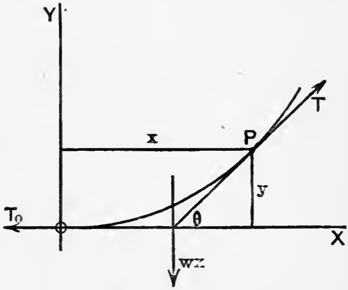

Fig. 10.

If we draw a triangle whose sides are parallel to these forces they are proportional to the forces, and if θ is the inclination of T to the horizontal

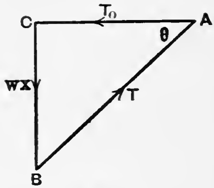

Fig. 11.

$$\frac{T_0}{T} = \cos\theta \text{..............(1)},$$

and

$$\frac{wx}{T_0} = \tan\theta \text{..............(2)},$$

but $\tan\theta$ is $\dfrac{dy}{dx}$, so that $\dfrac{dy}{dx} = \dfrac{w}{T_0}x$...(3);

hence, integrating, $$y = \frac{1}{2}\frac{w}{T_0}x^2 + \text{constant}.$$

Now we see that y is 0 when x is 0, so that the constant is 0. Hence the equation to the curve is

$$y = \frac{1}{2}\frac{w}{T_0}x^2 \text{ ....(4)},$$

and it is a parabola. Now $\tan\theta$ is $\frac{w}{T_0}x$, so that $\sec^2\theta$ is $1+\frac{w^2}{T_0^2}x^2$. And as $T=T_0\sec\theta$, $T=T_0\sqrt{1+\frac{w^2}{T_0^2}x^2}$ ...(5).

From this, all sorts of calculations may be made. Thus if l is the span and D the dip of a telegraph wire, if the whole curve be drawn it will be seen that we have only to put in (4) the information that when $x=\frac{1}{2}l$, $y=D$,

or
$$D=\frac{1}{2}\frac{w}{T_0}\frac{1}{4}l^2 \text{ or } T_0=\frac{wl^2}{8D},$$

and the greater tension elsewhere is easy to find.

In the problem of the shape of any uniform chain, loaded only with its own weight, the integration is not so easy. I give it in a note*. When it is so flat that we may take the

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* The integration in this note requires a knowledge of Chapter III.

If the weight of the portion of chain OP, instead of being wx is ws, where s is the length of the curve from O to P, the curve y is called **the Catenary.** Equation (3) above becomes

$$\frac{dy}{dx}=\frac{ws}{T_0}, \text{ or letting } T_0=wc,\ \frac{dy}{dx}=\frac{s}{c} \text{ ....(1).}$$

If δs is the length of an elementary bit of chain, we see that in the limit

$$(\delta s)^2=(\delta x)^2+(\delta y)^2$$

so that
$$\frac{ds}{dy}=\sqrt{\left(\frac{dx}{dy}\right)^2+1},$$

and hence $\frac{dy}{ds}=\frac{s}{\sqrt{c^2+s^2}}$. This being integrated gives $y+c=\sqrt{c^2+s^2}$ ...(2), the constant added in integration being such that $s=0$ when $y=0$. From (2) we find $s^2=y^2+2yc$...(3), and using this in (1), we have

$$\frac{dx}{dy}=\frac{c}{\sqrt{y^2+2yc}},$$

the integral of which is

$$x=c\log\frac{y+c+\sqrt{y^2+2yc}}{c},$$

as when $y=0$, $x=0$, if O is the origin, no constant is to be added. Putting this in the exponential form

$$c\epsilon^{x/c}=y+c+\sqrt{y^2+2yc},$$

transposing and squaring we find

$$y+c=\tfrac{1}{2}c\left(\epsilon^{x/c}+\epsilon^{-x/c}\right).$$

load on any piece of it as proportional to the horizontal projection of it, we have the parabolic shape. †

39. Efficiency of Heating Surface of Boiler. If 1 lb. of gases in a boiler flue would give out the heat θ in cooling to the temperature of the water (θ may be taken as proportional to the difference of temperature between gases and water, but this is not quite correct), we find from Peclet's experiments that the heat per hour that flows through a square foot of flue surface is, roughly, $m\theta^2$. Let $\theta = \theta_1$ at the furnace end of a flue and $\theta = \theta_2$ at the chimney end. Let us study what occurs at a place in the flue.

The gases having passed the area S in coming from the furnace to a certain place where the temperature is θ, pro-

† Or changing the origin to a point at the distance c below O, as at O in fig. 12 where SP is y' and RP is x, we have

$$y' = \tfrac{1}{2}c\left(\epsilon^{x/c} + \epsilon^{-x/c}\right) \quad \ldots\ldots\ldots (4).$$

This is sometimes called $\quad y' = c \cosh x/c.$

Using (1) we find $\quad s = \tfrac{1}{2}c\left(\epsilon^{x/c} - \epsilon^{-x/c}\right),$

sometimes called $\quad s = c \sinh x/c.$

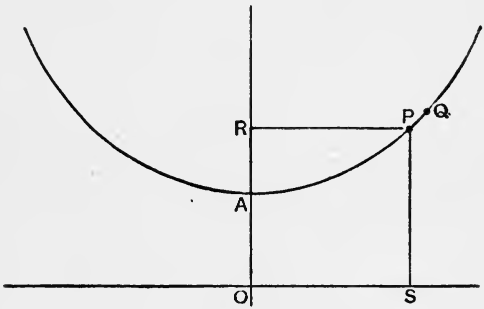

Fig. 12.

Note that tables of the values of sinh u and cosh u have been published. Returning to the original figure, the tension at P being T,

$$\frac{T}{ws} = \frac{AB}{BC} = \frac{ds}{dy}, \text{ and from (3), } s \cdot \frac{ds}{dy} = y + c,$$

so that $\quad \dfrac{T}{ws} = \dfrac{y+c}{s}$. Hence $T = w(y+c)$ or $T = wy'$.

ceed further on to a place where S has become $S+\delta S$ and θ has become $\theta+\delta\theta$ (really $\delta\theta$ is negative as will be seen). A steady state is maintained and during one hour the gases lose the heat $m\theta^2 . \delta S$ through the area δS. If during the hour W lb. of gases lost at the place the amount of heat $-W . \delta\theta$, then

$$-W . \delta\theta = m\theta^2 . \delta S,$$

or rather
$$\frac{dS}{d\theta} = -\frac{W}{m} \cdot \frac{1}{\theta^2} \quad \text{.......................(1).}$$

That is, integrating with regard to θ,

$$S = \frac{W}{m}\frac{1}{\theta} + c \quad \text{....................(2),}$$

where c is some constant.

Putting in $\theta = \theta_1$ the temperature at the furnace end when $S=0$, we have

$$0 = \frac{W}{m}\frac{1}{\theta_1} + c \text{ or } c = -\frac{W}{m}\frac{1}{\theta_1},$$

so that (2) becomes

$$S = \frac{W}{m}\left(\frac{1}{\theta} - \frac{1}{\theta_1}\right) \quad \text{....................(3).}$$

This shows how θ diminishes as S increases from the furnace end, and it is worth a student's while to plot the curve connecting S and θ. If now S is the whole area of heating surface and $\theta = \theta_2$ at the smoke-box end,

$$S = \frac{W}{m}\left(\frac{1}{\theta_2} - \frac{1}{\theta_1}\right) \quad \text{....................(4).}$$

The heat which one pound of gases has at the furnace end is θ_1, it gives up to the water the amount $\theta_1 - \theta_2$. Therefore the **efficiency** of the heating surface may be taken as

$$E = \frac{\theta_1 - \theta_2}{\theta_1} \quad \text{.......................(5),}$$

and it follows from (4) that

$$E = \frac{1}{1 + \dfrac{W}{\theta_1 m S}}.$$

Now if W' is the weight of coals burnt per hour; $W = 13W'$ if air is admitted just sufficient for complete combustion; $W =$ about $20W'$ in the case of ordinary forced draught; $W =$ about $26W'$ in the case of chimney draught. In these cases θ_1 does not seem to alter inversely as W, as might at first sight appear: but we do not know exactly how θ_1 depends upon the amount of excess of air admitted. We can only say that if $W' \div S$ is the weight of coal per hour per square foot of heating surface and we call it w, there seems to be some such law as $E = \dfrac{1}{1 + aw}$, where a depends upon the amount of air admitted. In practice it is found that $a = 0{\cdot}5$ for chimney draught and $0{\cdot}3$ for forced draught, give fairly correct results. Also the numerator may be taken as greater than 1 when there are special means of heating the feed water.

Instead of the law given above (the loss of heat by gases in a flue $\propto \theta^2$), if we take what is probably more likely, that the loss is proportional to θ,

Then (1) above becomes

$$\frac{dS}{d\theta} = -\frac{W}{m}\frac{1}{\theta} \quad\text{.......................(1)},$$

or

$$S = -\frac{W}{m}\log\theta + \text{constant} \quad\text{..............(2)}.$$

Let $\theta = \theta_1$ at furnace end or when $S = 0$ so that our constant is $\dfrac{W}{m}\log\theta_1$ and (2) becomes

$$S = \frac{W}{m}\log\left(\frac{\theta_1}{\theta}\right) \quad\text{.....................(3)}.$$

If S is the area of the whole flue and θ_2 is the temperature at the smoke-box end, then

$$S = \frac{W}{m}\log\frac{\theta_1}{\theta_2} \quad\text{.......................(4)},$$

$$e^{\frac{Sm}{W}} = \frac{\theta_1}{\theta_2}.$$

The efficiency $$E=\frac{\theta_1-\theta_2}{\theta_1} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

becomes $$E=1-\frac{\theta_2}{\theta_1}=1-e^{-\frac{Sm}{W}} \quad\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Or if w is the weight of fuel per square foot of heating surface as above (6) becomes

$$E=1-e^{-\frac{1}{aw}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

40. Work done by Expanding Fluid*. If p is the pressure and v the volume at any instant, of a fluid which has already done work W in expanding, one good definition of pressure is $p=\frac{dW}{dv}$ ... (1), or in words, pressure is the rate at which work is done per unit change of volume. Another way of putting this is; if the fluid expands through the volume δv there is an increment δW of work done so that $p\,.\,\delta v=\delta W$, or $p=\frac{\delta W}{\delta v}$, but this is only strictly true when δv is made smaller and smaller without limit, and so (1) is absolutely true. Now if the fluid expands according to the law $pv^s=c$, a constant ... (2); $p=cv^{-s}$, and this is the differential coefficient of W with regard to v or, as we had better write it down, $\frac{dW}{dv}=cv^{-s}$.

We therefore integrate it according to our rule and we have

$$W=\frac{+c}{-s+1}v^{-s+1}+C \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where C is some constant. To find C, let us say that we shall only begin to count W from $v=v_1$. That is, $W=0$ when $v=v_1$. Then

$$0=\frac{c}{1-s}v_1^{1-s}+C, \text{ so that } C=-\frac{c}{1-s}v_1^{1-s}.$$

Insert this value of C in (3) and we have

$$W=\frac{c}{1-s}(v^{1-s}-v_1^{1-s}) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

which is the work done in expanding from v_1 to v.

Now if we want to know W when $v=v_2$, we have

$$W_{12}=\frac{c}{1-s}(v_2^{1-s}-v_1^{1-s}) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

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* Observe that if for p and v we write y and x this work becomes very easy.

This answer may be put in other shapes. Thus from (2) we know that

$$c = p_1 v_1^s \text{ or } p_2 v_2^s,$$

so that
$$W_{12} = \frac{p_1 v_1^s}{1-s}(v_2^{1-s} - v_1^{1-s}),$$

or
$$W_{12} = \frac{p_1 v_1}{1-s}\left\{\left(\frac{v_2}{v_1}\right)^{1-s} - 1\right\},$$

or
$$W_{12} = \frac{p_1 v_1}{s-1}\left\{1 - \left(\frac{v_1}{v_2}\right)^{s-1}\right\} \quad \text{.....................(6)},$$

a formula much used in gas engine and steam engine calculations.

There is one case in which this answer turns out to be useless; try it when $s=1$. That is, find what work is done from v_1 to v_2 by a fluid expanding according to the law (it would be the isothermal law if the fluid were a gas)

$$pv = c.$$

If you have noticed how it fails, go back to the statement

$$\frac{dW}{dv} = cv^{-1} \quad \text{...............................(7)}.$$

You will find that when you integrate x^m with regard to m, the general answer has no meaning, cannot be evaluated, if $m=-1$. But I have already said, and I mean to prove presently that the integral of x^{-1} is $\log x$. So the integral of (7) is

$$W = c\log v + C.$$

Proceeding as before we find that, in this particular case,

$$W_{12} = c\log\frac{v_2}{v_1} \quad \text{.............................(8)}.$$

41. Hypothetical Steam Engine Diagram.

Let steam be admitted to a cylinder at the constant pressure p_1, the volume increasing from 0 to v_1 in the cylinder. The work done is $v_1 p_1$. Let the steam expand to the volume v_2 according to the law $pv^s = c$. The work done is given by (6) or (8). Let the back pressure be p_3, then the work done in driving out the steam in the back stroke is $p_3 v_2$. We neglect cushioning in this hypothetical diagram. Let $v_2 \div v_1$ be called r the ratio of cut-off. Then the nett work done altogether is

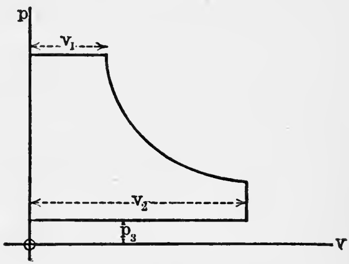

Fig. 13.

$$p_1 v_1 + \frac{p_1 v_1}{s-1}\{1 - r^{1-s}\} - p_3 v_2.$$

If p_e is the effective pressure so that $p_e v_2$ is equal to the above nett work (p_e is measured from actual indicator diagrams, as the average pressure); putting it equal and dividing by v_2 we have on simplifying

$$p_e = p_1 \frac{1}{r} + \frac{p_1}{s-1} \frac{1}{r} (1 - r^{1-s}) - p_3$$

$$= \frac{p_1}{s-1} \left(\frac{s}{r} - r^{-s}\right) - p_3.$$

In the special case of $s=1$ we find $p_e = p_1 \dfrac{1+\log r}{r} - p_3$ in the same way.

42. Definite Integral. *Definition.* The symbol $\int_b^a f(x)\,.\,dx$ tells us:—"Find the general integral of $f(x)$; insert in it the value a for x, insert in it the value b for x; subtract the latter from the former value*." This is said to be the

---

* The symbol $\int_a^b \int_{f(x)}^{F(x)} u\,.\,dx\,.\,dy$...(1), tells us to integrate u (which is a function of x and y), with regard to y, as if x were constant; then insert $F(x)$ for y and also $f(x)$ for y and subtract. This result is to be integrated with regard to x, and in the answer a and b are inserted for x and the results subtracted.

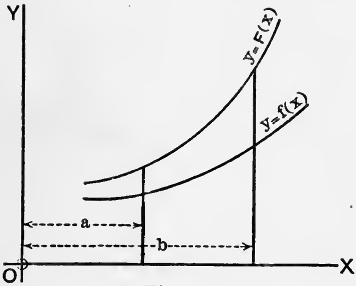

Fig. 14.

I. If $u=1$, $dx\,.\,dy$ evidently means an element of area, a little rectangle. The result of the first process leaves

$$\int_a^b \{F(x) - f(x)\}\,dx \text{ ......(2)}$$

still to be done. Evidently we have found the area included between the curves $y=F(x)$ and $y=f(x)$ and two ordinates at $x=a$ and $x=b$. Beginners had better always use form (2) in finding areas, see fig. 14.

II. If u is, say, the weight of gold per unit area upon the above mentioned area, then $u\,.\,dx\,.\,dy$ is the weight upon the little elementary area $dx\,.\,dy$, and our integral means the weight of all the gold upon the area I have mentioned.

When writers of books wish to indicate generally that they desire to integrate some property u (which at any place is a function of x, y, z), throughout some volume, they will write it with a triple integral,

$$\iiint u\,.\,dx\,.\,dy\,.\,dz,$$

and summation over a surface by $\iint v\,.\,dx\,.\,dy$.

integral of $f(x)$ between the limits a and b. Observe now that any constant which may be found in the general integral simply disappears in the subtraction.

In integrating between limits we shall find it convenient to work in the following fashion.

Example, to find $\int_b^a x^2 . dx$. The general integral is $\frac{1}{3}x^3$ and we write $\int_b^a x^2 . dx = \left[\begin{smallmatrix}a\\ \\ b\end{smallmatrix} \frac{1}{3}x^3\right] = \frac{1}{3}a^3 - \frac{1}{3}b^3$.

Symbolically. If $F(x)$ is the general integral of $f(x)$ then $\int_b^a f(x) . dx = \left[\begin{smallmatrix}a\\ \\ b\end{smallmatrix} F(x)\right] = F(a) - F(b)$.

Note as evidently true from our definition, that

$$\int_b^a f(x) . dx = -\int_a^b f(x) . dx,$$

and also that

$$\int_b^a f(x) . dx = \int_b^c f(x) . dx + \int_c^a f(x) . dx.$$

43. **Area of a curve.** Let y of the curve be known as some function of x and let PS be the curve. It is required to find the area $MPQT$.

Now if the area $MPQT$ be called A and $OT = x$, $QT = y$, $OW = x + \delta x$, $WR = y + \delta y$, and the area $MPRW$ be $A + \delta A$ then $\delta A =$ area $TQRW$.

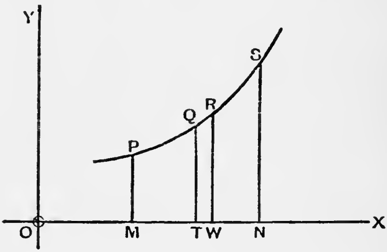

Fig. 15.

Indeed some writers use $\iint v . dS$ to mean generally the summation of v over an area, and $\int w . ds$ to mean the summation of w along a line or what is often called the line integral of w. The line integral of the pull exerted on a tram car means the work done. The surface integral of the normal velocity of a fluid over an area is the total volume flowing per second. Engineers are continually finding line, surface and volume integrals in their practical work and there is nothing in these symbols which is not already perfectly well known to them.

If the short distance QR were straight,

$$\delta A = \tfrac{1}{2}\delta x\,(TQ + RW) = \delta x\,(y + \tfrac{1}{2}\delta y).$$

Therefore $\dfrac{\delta A}{\delta x} = y + \tfrac{1}{2}\delta y$, as δx gets smaller and smaller and in the limit $\dfrac{dA}{dx} = y$ ....(1).

Hence A is such a function of x that y is its differential coefficient, or A is the integral of y.

In fig. 16 CQD is the curve $y = a + bx^2$ and $EMGF$ is the curve showing

$$A = C + ax + \tfrac{1}{3}bx^3,$$

so that A is the integral of y. In what sense does A represent the *area* of the curve CD? The ordinate of the A curve, GT, represents to some scale or other, the area of the y curve $MPQT$ from some standard ordinate MP.

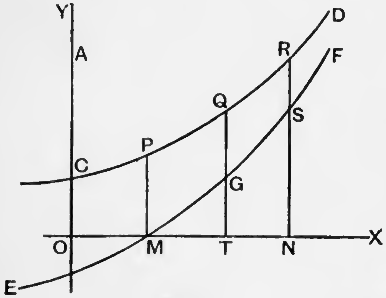

Fig. 16.

The ordinate TQ represents to scale, the slope of EF at G. Observe, however, that if we diminish or increase all the ordinates of the A curve by the same amount, we do not change its slope anywhere, and y, which is given us, only tells us the slope of A. Given the y curve we can therefore find any number of A curves; we settle the one wanted when we state that we shall reckon area from a particular ordinate such as MP. Thus, in fig. 16 if the general integral of y is $F(x) + c$. If we use the value $x = OM$ we have, area up to MP from some unknown standard ordinate $= F(OM) + c$.

Taking $x = ON$, we have area up to NR from some unknown standard ordinate $= F(ON) + c$. And the area between MP and NR is simply the difference of these $F(ON) - F(OM)$, the constant disappearing.

Now the symbol $\int_{OM}^{ON} y \, . \, dx$ tells us to follow these instructions:—integrate y; insert ON for x in the integral; insert OM for x in it; then subtract the latter. We see therefore

that the result of such an operation is the area of the curve between the ordinate at OM and the ordinate at ON.

If y and x represent any quantities whatsoever, and a curve be drawn with y as ordinate and x as abscissa, then the integral $\int y\,.\,dx$ is represented by the area of the curve, and we now know how to proceed when we desire to find the sum of all such terms as $y\,.\,\delta x$ between the limits $x=b$ and $x=a$ when δx is supposed to get smaller and smaller without limit.

Example. Find **the area** enclosed between the **parabolic curve** OA, the ordinate AB and the axis OB. Let the equation to the curve be

$$y=ax^{\frac{1}{2}}\,.........(1),$$

where $PQ=y$ and $OQ=x$. Let $QR=\delta x$.

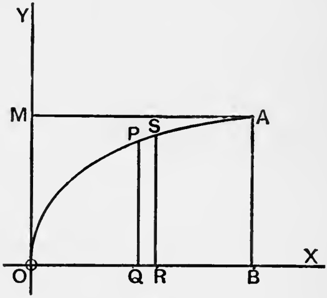

Fig. 17.

The area of the strip $PQRS$ is more and more nearly

$$ax^{\frac{1}{2}}\,.\,\delta x,$$

as δx is made smaller and smaller; or rather the whole area is $\int_0^{OB} ax^{\frac{1}{2}}\,.\,dx$, which is

$$a\left[\frac{2}{3}x^{1\frac{1}{2}}\right]_0^{OB}=\tfrac{2}{3}a\,.\,OB^{1\frac{1}{2}}\,.......................(2).$$

Now what is a in terms of AB and OB? When $y=AB$, $x=OB$. Hence by (1)

$$AB=a\,.\,OB^{\frac{1}{2}},\text{ so that } a=\frac{AB}{OB^{\frac{1}{2}}}.$$

$$\text{Therefore the area}=\frac{2}{3}\,\frac{AB}{OB^{\frac{1}{2}}}\,OB^{\frac{3}{2}}=\tfrac{2}{3}AB\,.\,OB\,;$$

that is, $\frac{2}{3}$rds of the area of the rectangle $OMAB$.

Observe that the area of a very flat segment of a circle is like that of a parabola when OB is very small compared with BA.

Exercise 1. Find **the area** between the curve $\mathbf{y = mx^{-n}}$ and the two ordinates at $x = a$ and $x = b$.

The answer is

$$\int_a^b mx^{-n} \cdot dx = \frac{m}{1-n}\left[\begin{matrix} {}^b \\ {}_a \end{matrix} x^{1-n}\right] = \frac{m}{1-n}(b^{1-n} - a^{1-n}).$$

Observe (as in Art. 40) that this fails when $n = 1$; that is, in the rectangular hyperbola.

In this case the answer is

$$m\int_a^b \frac{1}{x} \cdot dx = m\left[\begin{matrix} {}^b \\ {}_a \end{matrix} \log x\right] = m \log \frac{b}{a}.$$

The equation to any curve being

$$y = a + bx + cx^2 + ex^3 + fx^4,$$

the area is $\quad A = ax + \frac{1}{2}bx^2 + \frac{1}{3}cx^3 + \frac{1}{4}ex^4 + \frac{1}{5}fx^5.$

Here the area up to an ordinate at x is really measured from the ordinate where $x = 0$, because $A = 0$ when $x = 0$. We can at once find the area between any two given ordinates.

Exercise 2. Find the area of the curve $y = a\sqrt[3]{x}$ between the ordinates at $x = \alpha$ and $x = \beta$.

$$a\int_\alpha^\beta x^{\frac{1}{3}} \cdot dx = a\left[\begin{matrix} {}^\beta \\ {}_\alpha \end{matrix} \tfrac{3}{4}x^{\frac{4}{3}}\right] = \frac{3a}{4}(\beta^{\frac{4}{3}} - \alpha^{\frac{4}{3}}).$$

Exercise 3. Find the area of the curve $yx^2 = a$ between the ordinates at $x = \alpha$ and $x = \beta$.

$$\text{Answer: } a\int_\alpha^\beta x^{-2} \cdot dx = a\left[\begin{matrix} {}^\beta \\ {}_\alpha \end{matrix} - x^{-1}\right] = a(\alpha^{-1} - \beta^{-1}).$$

44. **Work done by Expanding Fluid.** When we use definite integrals the work is somewhat shorter than it was in Art. 40. For if $p = cv^{-s}$, the work done from volume v_1 to volume v_2 is

$$\int_{v_1}^{v_2} cv^{-s} \cdot dv \text{ or } c\left[\begin{matrix} {}^{v_2} \\ {}_{v_1} \end{matrix} \frac{1}{1-s}v^{1-s}\right] \text{ or } \frac{c}{1-s}(v_2^{1-s} - v_1^{1-s})$$

The method fails when $s = 1$ and then the integral is

$$c\left[\begin{matrix} {}^{v_2} \\ {}_{v_1} \end{matrix} \log_\epsilon v\right] = c \log_\epsilon \frac{v_2}{v_1}.$$

45. Centre of Gravity. Only a few bodies have centres of gravity. We usually mean the centre of mass of a body or the centre of an area.

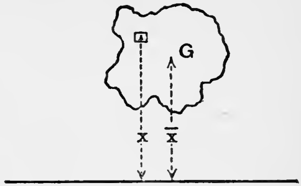

Fig. 18.

If each little portion of **a mass** be multiplied by its distance from any plane, and the results added together, they are equal to the whole mass multiplied by the distance of its centre, $\bar{x}$, from the same plane. Expressed algebraically this is

$$\Sigma mx = \bar{x}\Sigma m.$$

If each little portion of a plane **area**, as in fig. 18, be multiplied by its distance from any line in its plane and the results added together, they are equal to the whole area multiplied by the distance of its centre $\bar{x}$ from the same line. Expressed algebraically this is $\Sigma ax = \bar{x}\Sigma a$.

Example. Find the centre of mass of a right **cone.** It is evidently in the axis OB of the cone. Let the line OA rotate about OX, it will generate a cone. Consider the circular slice PQR of thickness δx. Let $OQ = x$, then PQ or

$$y = \frac{AB}{OB}x.$$

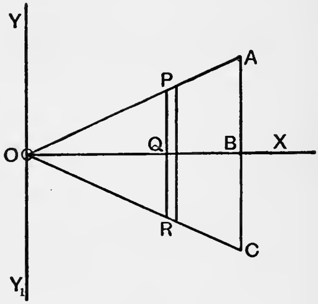

Fig. 19.

The mass of PR multiplied by the distance from O to its centre is equal to the sum of the masses of all its parts each multiplied by its distance from the plane YOY_1. The volume of the slice PR being its area πy^2 multiplied by its thickness δx; multiply this by m the mass per unit volume and we have its mass $m\pi y^2 . \delta x$. As the slice gets thinner and thinner, the distance of its centre from O gets more and more nearly x. Hence we have to find the sum of all such terms as $m\pi xy^2 . \delta x$, and put it equal to the whole mass $(\frac{1}{3}\pi m . AB^2 . OB)$ multiplied by $\bar{x}$, the distance of

its centre of gravity from O. Putting in the value of y^2 in terms of x we have

$$\int_0^{OB} m\pi \left(\frac{AB}{OB}\right)^2 x^3 \,.\, dx \text{ equated to } \tfrac{1}{3}\pi m AB^2 \,.\, OB \,.\, \bar{x}.$$

Now $$\int_0^{OB} x^3 \,.\, dx = \left[\tfrac{1}{4}x^4\right]_0^{OB} = \tfrac{1}{4}OB^4,$$

and hence $m\pi \left(\dfrac{AB}{OB}\right)^2 \tfrac{1}{4}OB^4 = \tfrac{1}{3}\pi m \,.\, AB^2 \,.\, OB \,.\, \bar{x}.$

Hence $\bar{x} = \frac{3}{4}OB$. That is, the centre of mass is $\frac{3}{4}$ of the way along the axis from the vertex towards the base.

46. It was assumed that students knew how to find **the volume of a cone.** We shall now prove the rule.

The volume of the slice PR is $\pi \,.\, y^2 \,.\, \delta x$ and the whole volume is

$$\int_0^{OB} \pi y^2 \,.\, dx = \int_0^{OB} \pi \left(\frac{AB}{OB}\right)^2 x^2 \,.\, dx = \pi \left(\frac{AB}{OB}\right)^2 \left[\tfrac{1}{3}x^3\right]_0^{OB}$$

$$= \pi \left(\frac{AB}{OB}\right)^2 \tfrac{1}{3} \,.\, OB^3 = \tfrac{1}{3}\pi \,.\, AB^2 \,.\, OB,$$

or $\frac{1}{3}$ of the volume of a cylinder on the same base AC and of the same height OB. If we had taken $y = ax$ all the work would have looked simpler.

Example. **Find the volume and centre of mass** of uniform material (of mass m per unit volume) bounded by a **paraboloid of revolution.**

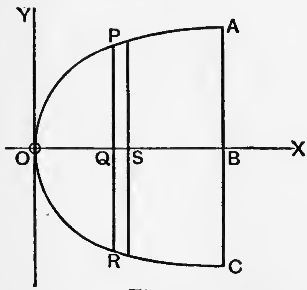

Fig. 20.

Let $PQ = y$, $OQ = x$, $QS = \delta x$.

Let the equation to the curve OPA be $y = ax^{\frac{1}{2}}$ ....(1).

The volume of the slice PSR is $\pi y^2 \,.\, \delta x$; so that the whole volume is $\int_0^{OB} \pi \,.\, a^2 x \,.\, dx$ or

$\frac{1}{2}\pi a^2 \,.\, OB^2$ ......(2).

Now what is a? When

$$y = AB,\ x = OB,$$

so that from (1), $AB = a \,.\, OB^{\frac{1}{2}}$ and a is $\dfrac{AB}{OB^{\frac{1}{2}}}$. Hence the volume is $\frac{1}{2}\pi \dfrac{AB^2}{OB} OB^2$ or

$$\tfrac{1}{2}\pi \,.\, AB^2 \,.\, OB \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

That is, half the area of the circle AC multiplied by the height OB. Hence the volume of the paraboloid is half the volume of a cylinder on the same base and of the same height. (The volumes of Cylinder, Paraboloid of revolution, and Cone of same bases and heights are as $1 : \frac{1}{2} : \frac{1}{3}$.)

Now as to the centre of mass of the Paraboloid. It is evidently on the axis. We must find $\int_0^{OB} m\pi \,.\, y^2 x \,.\, dx$, or

$$\int m\pi x \,.\, a^2 x \,.\, dx, \text{ or } m\pi a^2 \int x^2 \,.\, dx,$$

or $m\pi a^2 \,.\, \left[\tfrac{1}{3}x^3\right]_0^{OB}$ and this is $\frac{1}{3} m\pi a^2 \,.\, OB^3$. Inserting as before the value of a^2 or $\dfrac{AB^2}{OB}$ we have the integral equal to $\frac{1}{3}m\pi . OB^2 . AB^2$. This is equal to the whole mass multiplied by the x of the centre of mass, $\bar{x}$, or $m\frac{1}{2}\pi \,.\, AB^2 \,.\, OB \,.\, \bar{x}$, so that $\bar{x} = \frac{2}{3}OB$. The centre of mass of a paraboloid of revolution is $\frac{2}{3}$rds of the way along the axis towards the base from the vertex.

Example. The curve $\mathbf{y = ax^n}$ revolves about the axis of x, find the volume enclosed by **the surface of revolution** between $x = 0$ and $x = b$.

The volume of any surface of revolution is obtained by integrating $\pi y^2 \,.\, dx$. Hence our answer is

$$\pi \int_0^b a^2 x^{2n} \,.\, dx = \frac{\pi a^2}{2n+1} \left[x^{2n+1}\right]_0^b = \frac{\pi a^2}{2n+1} b^{2n+1}.$$

Find its centre of mass if m is its mass per unit volume. For any solid of revolution we integrate $m \,.\, x\pi y^2 \,.\, dx$ and

divide by the whole mass which is the integral of $m\pi y^2 dx$. If m is constant we have

$$m\pi\int_0^b x a^2 x^{2n} \cdot dx = m\pi a^2 \int_0^b x^{2n+1} dx$$

$$= \frac{m\pi a^2}{2n+2}\left[{}_0^b x^{2n+2}\right] = \frac{m\pi a^2}{2n+2} b^{2n+2},$$

and the whole mass is $\dfrac{m\pi a^2}{2n+1} b^{2n+1}$, so that $\bar{x} = \dfrac{2n+1}{2n+2} b$.

Suppose m is not constant but follows the law

$$m = m_0 + cx^s.$$

To find the mass and centre of mass of the above solid. Our first integral is

$$\pi\int(m_0 x + cx^{s+1})\, a^2 x^{2n} \cdot dx, \text{ or } a^2\pi\int(m_0 x^{2n+1} + cx^{2n+s+1})\, dx,$$

or $$a^2\pi\left[{}_0^b \frac{m_0}{2n+2} x^{2n+2} + \frac{c}{2n+s+2} x^{2n+s+2}\right] \ldots\ldots\ldots(1).$$

The mass is $a^2\pi\displaystyle\int_0^b (m_0 + cx^s)\, x^{2n} \cdot dx$

or $$a^2\pi\left[\frac{m_0}{2n+1} x^{2n+1} + \frac{c}{2n+s+1} x^{2n+s+1}\right] \ldots\ldots\ldots\ldots(2).$$

Substituting b for x in both of these and dividing (1) by (2), we find $\bar{x}$.

An ingenious student can manufacture for himself many exercises of this kind which only involve the integration of x^n.

An arc of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ revolves about the axis of x, find the volume of the portion of the **ellipsoid of revolution** between the two planes where $x = 0$ and where $x = c$. Here $y^2 = \dfrac{b^2}{a^2}(a^2 - x^2)$. The integral of πy^2 is

$$\pi\frac{b^2}{a^2}\left[{}_0^c a^2 x - \tfrac{1}{3}x^3\right] = \pi\frac{b^2}{a^2}(a^2 c - \tfrac{1}{3}c^3).$$

The volume of the whole ellipsoid is $\frac{4\pi}{3}b^2a$ and of a sphere it is $\frac{4\pi}{3}a^3$.

47. Lengths of Curves. In fig. 21 the co-ordinates of P are x and y and of Q they are $x+\delta x$ and $y+\delta y$. If we call the length of the curve from some fixed place to P by the name s and the length PQ, δs, then $(\delta s)^2=(\delta x)^2+(\delta y)^2$ more and more nearly as δx gets smaller, so that

$$\frac{\delta s}{\delta x}=\sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2},$$

or rather, in the limit

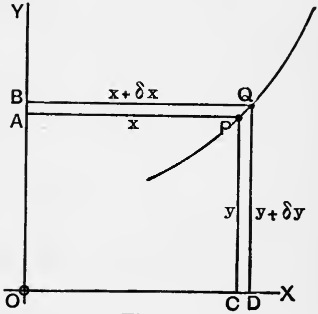

Fig. 21.

$$\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

To find s then, we have only to integrate $\sqrt{1+\left(\frac{dy}{dx}\right)^2}$.

It is unfortunate that we are only supposed to know as yet $\int x^n \, . \, dx$, because this does not lend itself much to exercises on the lengths of curves.

Example. Find the length of the curve $y=a+bx$ (a straight line) between the limits $x=0$ and $x=c$.

Here $\frac{dy}{dx}=b$ and $\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+b^2}$,

$$s=\int_0^c \frac{ds}{dx}\, . \, dx=\int_0^c \sqrt{1+b^2}\, . \, dx=\left[x\sqrt{1+b^2}\right]_0^c$$

$$=c\sqrt{1+b^2}.$$

Exercise. There is a curve whose slope is $\sqrt{a^2x^n - 1}$, find an expression for its length. Answer: $s = \dfrac{2a}{n+2} x^{(n+2)/2}$.

Other exercises on lengths of curves will be given later.

48. Areas of Surfaces of Revolution. When the curve APB revolving about the axis OX describes a surface of revolution, we have seen that the volume between the ends ACA' and BDB' is the integral of πy^2 with regard to x between the limits OC and OD.

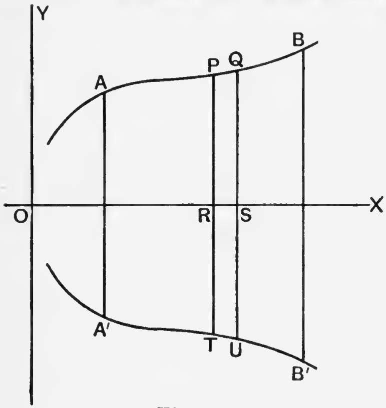

Fig. 22.

Again the elementary *area* of the surface is what is traced out by the elementary length PQ or δs and is in the limit $2\pi y \,.\, ds$. Hence we have to integrate $\int_{OC}^{OD} 2\pi y \,.\, \frac{ds}{dx} \,.\, dx$, and as the law of the curve is known, $y \,.\, \frac{ds}{dx}$ or $y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ can be expressed in terms of x.

Example. The line $y = a + bx$ revolves about the axis of x; find the surface of the cone between the limits $x = 0$ and $x = c$.

$\frac{dy}{dx} = b$, so that the area is $2\pi \int_0^c y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \,.\, dx$

$$= 2\pi\sqrt{1+b^2}\int_0^c (a+bx)\,dx = 2\pi\sqrt{1+b^2}\left[_0^c\, ax + \tfrac{1}{2}bx^2\right]$$

$$= 2\pi\sqrt{1+b^2}\,(ac + \tfrac{1}{2}bc^2).$$

The problem of finding the area of a spherical surface is here given in small printing because the beginner is supposed to know only how to differentiate x^n and this problem requires him to know that the differential coefficient of y^2

with regard to x is the differential coefficient with regard to y multiplied by $\frac{dy}{dx}$, or $2y \cdot \frac{dy}{dx}$. As a matter of fact this is not a real difficulty to a thinking student. The student can however find the area in the following way. Let V be the volume of the sphere of radius r, $V = \frac{4\pi}{3} r^3$, Art. 46. Let $V + \delta V$ be the volume of a sphere of radius $r + \delta r$, then

$$\delta V = \delta r \cdot \frac{dV}{dr} = \delta r\,(4\pi r^2),$$

which is only true when δr is supposed to be smaller and smaller without limit. Now if S is the surface of the spherical shell of thickness δr, its volume is $\delta r \cdot S$. Hence $\delta r \cdot S = \delta r \cdot 4\pi r^2$ and hence the area of a sphere is $4\pi r^2$.

Example. Find the area of the surface of a sphere. That is, imagine the quadrant of a circle AB of radius a, fig. 23, to

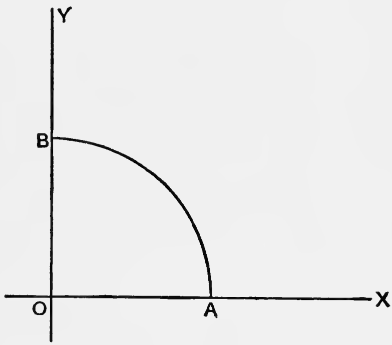

Fig. 23.

revolve about OX and take double the area generated. We have as the area, $4\pi \int_0^a y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$.

In the circle $\quad x^2 + y^2 = a^2$, or $y = \sqrt{a^2 - x^2}$,

$$2x + 2y \cdot \frac{dy}{dx} = 0, \text{ or } \frac{dy}{dx} = -\frac{x}{y}.$$

Hence as
$$1+\left(\frac{dy}{dx}\right)^2=1+\frac{x^2}{y^2}=\frac{a^2}{y^2},$$

$$4\pi\int_0^a y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,.\,dx=4\pi\int_0^a c\,.\,dx=4\pi\left[{}_0^a\;ax\right]=4\pi a^2.$$

49. If each elementary portion δs of the length of a curve be multiplied by x its distance from a plane (if the curve is all in one plane, x may be the distance to a line in the plane) and the sum be divided by the whole length of the curve, we get the $\bar{x}$ of the centre of the curve, or as it is sometimes called, the centre of gravity of the curve. Observe that the centre of gravity of an area is not necessarily the same as the centre of gravity of the curved boundary.

Guldinus's Theorems. I. Volume of a Ring. BC, fig. 24, is any plane area; if it revolves about an axis OO lying in its own plane it will generate a ring. The volume of this ring is equal to the area of BC multiplied by the circumference of the circle passed through by the centre of area of BC.

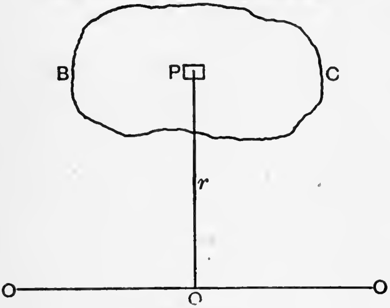

Fig. 24.

Imagine an exceedingly small portion of the area a at a place P at the distance r from the axis, the volume of the elementary ring generated by this is $a\,.\,2\pi r$ and the volume of the whole ring is the sum of all such terms or $V=2\pi\Sigma ar$. But $\Sigma ar=\bar{r}A$, if A is the whole area of BC. The student must put this in *words* for himself; $\bar{r}$ means the r of the centre of the area. Hence $V=2\pi\bar{r}\times A$ and this proves the proposition.

II. Area of a Ring. The area of the ring surface is the length of the Perimeter or boundary of BC multiplied by the circumference of the circle passed through by the centre of gravity of the boundary.

Imagine a very short length of the boundary, say δs, at

the distance r from the axis; this generates a strip of area of the amount $\delta s \times 2\pi r$. Hence the whole area is $2\pi\Sigma\delta s\,.\,r$. But $\Sigma\delta s\,.\,r = \bar{r} \times s$ if $\bar{r}$ is the distance of the centre of gravity of the boundary from the axis and s is the whole length of the boundary. Hence the whole area of the ring is $2\pi\bar{r} \times s$.

Example. Find the area of an anchor ring whose section is a circle of radius a, the centre of this circle being at the distance R from the axis. Answer:—the perimeter of the section is $2\pi a$ and the circumference of the circle described by its centre is $2\pi R$, hence the area is $4\pi^2 aR$.

Exercise. Find the volume and area of the rim of a fly-wheel, its mean radius being 10 feet, its section being a square whose side is 1·3 feet. Answer:

Volume $= (1{\cdot}3)^2 \times 2\pi \times 10$; Area $= 4 \times 1{\cdot}3 \times 2\pi \times 10$.

50. If every little portion of a mass be multiplied by the square of its distance from an axis, the sum is called the **moment of inertia of the whole mass** about the axis.

It is easy to prove that the moment of inertia about any axis is equal to the moment of inertia about a parallel axis through the centre of gravity together with the whole mass multiplied by the square of the distance between the two axes. Thus, let the plane of the paper be at right angles to the axes. Let there be a little mass m at P in the plane of the paper. Let O be the axis through the centre of gravity and O' be the other axis. We want the sum of all such terms as $m\,.\,(O'P)^2$.

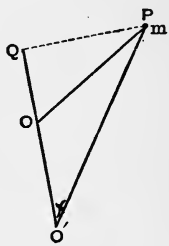

Fig. 25.

Now $(O'P)^2 = (O'O)^2 + OP^2 + 2\,.\,OO'\,.\,OQ$, where Q is the foot of a perpendicular from P upon OO', the plane containing the two axes. Then calling $\Sigma m\,.\,(O'P)^2$ by the name I, calling $\Sigma m\,.\,OP^2$ by the name I_0, the moment of inertia about the axis O through the centre of gravity of the whole mass, then, $I = (O'O)^2\Sigma m + I_0 + 2\,.\,OO'\,.\,\Sigma m\,.\,OQ$. But $\Sigma m\,.\,OQ$ means that each portion of mass m is multiplied by its distance from a plane at right angles to the paper through

the centre of gravity, and this must be 0 by Art. 45. So that the proposition is proved. Or letting Σm be called M the whole mass

$$\mathbf{I = I_0 + M \cdot (O'O)^2.}$$

Find the **moment of inertia of a circular cylinder** of length l about its axis. Let fig. 26 be a section, the axis being OO. Consider an elementary ring shown in section at $TQPR$ of inside radius r, its outside radius being $r+\delta r$. Its sectional area is $l\,.\,\delta r$

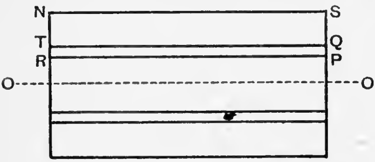

Fig. 26.

so that the volume of the ring is $2\pi r\,.\,l\,.\,\delta r$ and its mass is $m 2\pi r l\,.\,\delta r$. Its moment of inertia about OO is $2\pi m l\,.\,r^3\,.\,dr$ and this must be integrated between the limits $r=R$ the outside radius and $r=0$ to give the moment of inertia of the whole cylinder. The answer is $I_0 = \frac{1}{2}\pi m l R^4$. The whole mass $M = m l \pi R^2$. So that $I_0 = \dfrac{MR^2}{2}$. If we define the **radius of gyration** as k, which is such that $Mk^2 = I_0$, we have here $k^2 = \frac{1}{2}R^2$ or $k = \dfrac{1}{\sqrt{2}}R$.

The moment of inertia about the axis NS is

$$I = I_0 + M\,.\,R^2 = \tfrac{3}{2}MR^2,$$

so that the radius of gyration about NS is $R\sqrt{\frac{3}{2}}$.

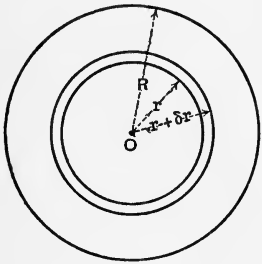

Fig. 27.

Moment of inertia of a **circle about its centre.** Fig. 27. Consider the ring of area between the circles of radii r and $r+\delta r$, its area is $2\pi r\,.\,\delta r$, more and more nearly as δr is smaller and smaller. Its moment of inertia is $2\pi r^3\,.\,dr$ and the integral of this between 0 and R is $\frac{1}{2}\pi R^4$ where R is the radius of the circle. The square of the radius of gyration is $\frac{1}{2}\pi R^4 \div$ the area $= \dfrac{R^2}{2}$

At any point O in an area, fig. 28, draw two lines OX and OY at right angles to one another. Let an elementary area a be at a distance x from one of the lines and at a distance y from the other and at a distance r from O. Observe that $ax^2 + ay^2 = ar^2$, so that if the moments of inertia of the whole area about the two lines be added together the sum is the moment of inertia about the point O. Hence the moment of inertia of a circle about a diameter is half the above, or $\frac{1}{4}\pi R^4$. The square of its radius of gyration is $\frac{1}{4}R^2$.

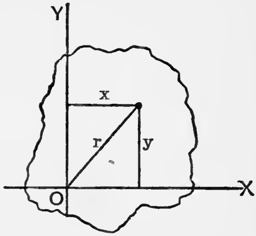

Fig. 28.

The **moment of inertia of an ellipse** about a principal diameter AOA. Let $OA = a$, $OB = b$.

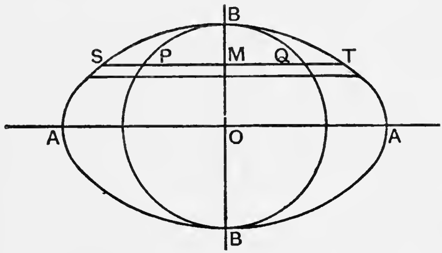

Fig. 29.

The moment of inertia of each strip of length ST is $\frac{a}{b}$ times the moment of inertia of each strip PQ of the circle, because it is at the same distance from AOA and $\frac{MT}{MQ} = \frac{a}{b}$. This is a property of ellipse and circle well known to all engineers. But the moment of inertia of the circle of radius b about AOA is $\frac{1}{4}\pi b^4$, so that the moment of inertia of the ellipse about AOA is $\frac{1}{4}\pi b^3 a$. Similarly its moment of inertia about BOB is $\frac{1}{4}\pi a^3 b$.

The above is a mathematical device requiring thought, not practical enough perhaps for the engineer's every-day work; it is given because we have not yet reached the inte-

gral which is needed in the straightforward working. The integral is evidently this. The area of the strip of length ST and breadth δy is $2x \,.\, \delta y$, the equation to the ellipse being $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, so that $x = \frac{a}{b}\sqrt{b^2 - y^2}$.

Then $2\int_0^b y^2 \,.\, 2x \,.\, dy$ or $4\frac{a}{b}\int_0^b y^2\sqrt{b^2 - y^2} \,.\, dy = I$.

The student ought to return to this as an example in Chap. III.

51. Moment of Inertia of Rim of Fly-wheel. If the rim of a fly-wheel is like a hollow cylinder of breadth l, the inside and outside radii being R_1 and R_2, the moment of inertia is $2\pi ml\int_{R_1}^{R_2} r^3 \,.\, dr$ or $2\pi ml\left[\frac{R^4}{4}\right]_{R_1}^{R_2} = \frac{1}{2}\pi ml(R_2^4 - R_1^4)$. The mass is $\pi(R_2^2 - R_1^2)\,lm = M$ say, so that $I = \frac{1}{2}(R_2^2 + R_1^2)M$. The radius of gyration is $\sqrt{\frac{1}{2}(R_2^2 + R_1^2)}$. It is usual to calculate the moment of inertia of the rim of a fly-wheel as if all its mass resided at the mean radius of the rim or $\frac{R_2 + R_1}{2}$. The moment of inertia calculated in this way is to the true moment of inertia as $\frac{(R_2 + R_1)^2}{2\,(R_2^2 + R_1^2)}$. Thus if $R_2 = R + a$, $R_1 = R - a$, the pretended I divided by the true I is $1 \div \left(1 + \frac{a^2}{R^2}\right)$ and if a is small, this is $1 - \frac{a^2}{R^2}$ nearly. If the whole mass of a fly-wheel, including arms and central boss, be M, there is usually no very great error in assuming that its moment of inertia is $I = R^2 M$.

52. A **rod** so thin that its thickness may be neglected is of length l, its mass being m per unit length, what is its moment of inertia about an axis at right angles to it, through one end O? Let x be the distance of a point from one end. An elementary portion of length δx of mass $m \,.\, \delta x$ has a moment of inertia $x^2 \,.\, m \,.\, dx$ and the integral of this from $x = 0$ to $x = l$ is $\frac{1}{3}ml^3$, which is the answer. As ml is the whole mass, the square of the radius of gyration is $\frac{1}{3}l^2$. I_0 the

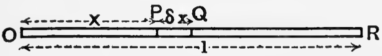

Fig. 30.

moment of inertia about a parallel axis through the middle of the rod, at right angles to its length, is

$$\tfrac{1}{3}ml^3 - ml \cdot \left(\frac{l}{2}\right)^2 \text{ or } ml^3\left(\tfrac{1}{3}-\tfrac{1}{4}\right) \text{ or } \tfrac{1}{12}ml^3.$$

So that the square of *this* radius of gyration is $\frac{1}{12}l^2$.

We shall now see what error is involved in neglecting the thickness of a cylindric rod.

If OO is an axis in the plane of the paper at right angles to the axis of a circular cylinder, through one end, and OP is x and R is the radius of the cylinder, its length being l; if ρ is the mass of the cylinder per unit volume; the moment of inertia about OO of the disc of radius R and thickness δx is $\pi R^2 \rho \delta x \cdot x^2 +$ the moment of inertia of the disc about its own diameter. Now we saw that the radius of gyration of a circle about its diameter was $\frac{R}{2}$, and the radius of gyration of the disc is evidently the same. Hence its moment of inertia about its diameter is $\frac{1}{4}R^2 \pi R^2 \cdot \delta x \cdot \rho$, or $\frac{1}{4}\pi\rho R^4 \cdot \delta x$. Hence the moment of inertia of the disc about O is

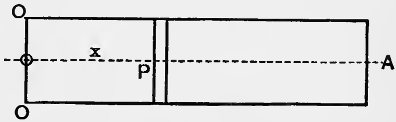

Fig. 31.

$$\pi R^2 \rho \left(x^2 \cdot \delta x + \tfrac{1}{4}R^2 \cdot dx\right).$$

If OA, the length of the rod, is l, we must integrate between 0 and l, and so we find

$$I = \pi R^2 \rho \left(\frac{l^3}{3} + \tfrac{1}{4}R^2 l\right).$$

The mass m per unit length is $\pi R^2 \rho$, so that

$$I = m\left(\frac{l^3}{3} + \tfrac{1}{4}R^2 l\right), \text{ or } \frac{ml^3}{3}\left(1 + \frac{3}{4}\frac{R^2}{l^2}\right),$$

$$I_0 = m\left(\frac{l^3}{12} + \tfrac{1}{4}R^2 l\right).$$

This is the moment of inertia about an axis through the centre of gravity parallel to OO.

53. *Example.* Where is the **Centre of Area** of the parabolic segment shown in fig. 20? The whole area is

$$\tfrac{2}{3}AC \times OB.$$

The centre of area is evidently in the axis.

The area of a strip PSR is $2y \,.\, \delta x$ and we must integrate $2xy \,.\, \delta x$. Now $y = ax^{\frac{1}{2}}$ where $a = AB \div OB^{\frac{1}{2}}$.

Hence $2\int_0^{OB} x \frac{AB}{OB^{\frac{1}{2}}} x^{\frac{1}{2}} \,.\, dx = \frac{2}{3} AC \,.\, OB \,.\, \bar{x}$. The integral is $2\frac{AB}{OB^{\frac{1}{2}}}\left[\frac{2}{5} x^{\frac{5}{2}}\right]_0^{OB}$ or $\frac{4}{5}\frac{AB}{OB^{\frac{1}{2}}} OB^{\frac{5}{2}}$, so that $\frac{4}{5} AB \,.\, OB^2 = \frac{4}{3} AB \,.\, OB \,.\, \bar{x}$ or $\bar{x} = \frac{3}{5} OB$.

Find the centre of area of the segment of the symmetrical area bounded by $\pm y = ax^n$ between $x = b$ and $x = c$.

We must divide the integral $2\int_b^c x \,.\, ax^n \,.\, dx$ by the area $2\int_b^c ax^n \,.\, dx$.

Or $$2a\left(\frac{c^{n+2} - b^{n+2}}{n+2}\right) \div 2a\left(\frac{c^{n+1} - b^{n+1}}{n+1}\right) = \bar{x},$$

$$\bar{x} = \frac{c^{n+2} - b^{n+2}}{c^{n+1} - b^{n+1}} \cdot \frac{n+1}{n+2}.$$

Many interesting cases may be taken. Observe that if the dimensions of the figure be given, as in fig. 20: thus if AB and PQ and BQ are given, we may find the position of the centre of the area in terms of these magnitudes.

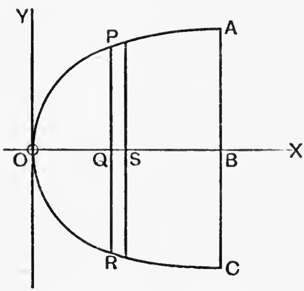

Fig. 32.

54. Moment of Inertia of a Rectangle.

The moment of inertia of a rectangle about the line OO

through its centre, parallel to one side. Let $AB = b$, $BC = d$.

Consider the strip of area between $OP = y$ and $OQ = y + \delta y$. Its area is $b \,.\, \delta y$ and its moment of inertia about OO is $b \,.\, y^2 \,.\, \delta y$, so that the moment of inertia of the whole rectangle is

$$b \int_{-\frac{1}{2}d}^{\frac{1}{2}d} y^2 \,.\, dy \text{ or } b \left[\tfrac{1}{3}y^3\right]_{-\frac{1}{2}d}^{\frac{1}{2}d} \text{ or } \frac{bd^3}{12}.$$

This is the moment of inertia which is so important in calculations on beams.

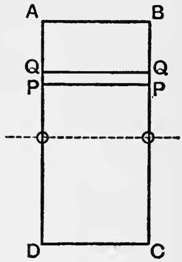

Fig. 33.

55. **Force of Gravity.** A uniform spherical shell of attracting matter exercises no force upon a body inside it. On unit mass outside, it acts as if all its mass were gathered at its centre.

The earth then exercises a force upon unit mass at any point P outside it which is inversely proportional to the square of r the distance of P from the centre. But if P is inside the earth, the attraction there upon unit mass is the mass of the sphere *inside* P divided by the square of r.

1. If the earth were homogeneous. If m is the mass per unit volume and R is the radius of the earth, the attraction on any outside point is $\frac{4\pi}{3} mR^3 \div r^2$.

The attraction on any inside point is $\frac{4\pi}{3} mr^3 \div r^2$ or $\frac{4\pi}{3} mr$. The attraction then at the surface being called 1, at any outside point it is $R^2 \div r^2$ and at any inside point it is $r \div R$. Students ought to illustrate this by a diagram.

2. If m is greater towards the centre, say $m = a - br$, then as the area of a shell of radius r is $4\pi r^2$, its mass is $4\pi r^2 \,.\, m \,.\, \delta r$, so that the whole mass of a sphere of radius r is $4\pi \int_0^r r^2 (a - br)\, dr$, or $\frac{4\pi}{3} ar^3 - \pi br^4$. Hence on any inside point the attraction is $\frac{4\pi}{3} ar - \pi br^2$ and on any outside point it is $\left(\frac{4\pi}{3} aR^3 - \pi bR^4\right) \Big/ r^2$.

Dividing the whole mass of the earth by its volume $\frac{4\pi}{3}R^3$, we find its mean density to be $a - \frac{3}{4}bR$, and the ratio of its mean density to the density at the surface is

$$(4a - 3bR)/(4a - 4bR).$$

56. Strength of thick Cylinders.

The first part of the following is one way of putting the well known theory of what goes on in a thin cylindric shell of a boiler. It prevents trouble with + and − signs afterwards, to imagine the fluid pressure to be greater outside than inside and the material to be in compression.

Consider the elementary thin cylinder of radius r and of thickness δr. Let the pressure inside be p and outside $p + \delta p$ and let the crushing stress at right angles to the radii in the material be q. Consider the portion of a ring $PQSR$ which is of unit length at right angles to the paper.

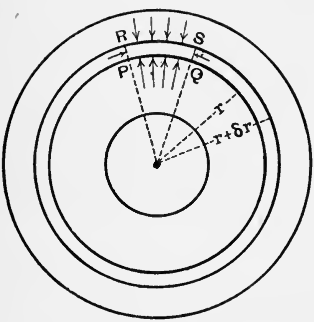

Fig. 34.

Radially we have $p + \delta p$ from outside acting on the area RS or $(r + \delta r)\,\delta\theta$ if $QOP = \delta\theta$, because the arc RS is equal to radius multiplied by angle; and $p \,.\, r \,.\, \delta\theta$ from inside or

$$(p + \delta p)(r + \delta r)\,\delta\theta - pr \,.\, \delta\theta$$

is on the whole the radial force from the outside more and more nearly as $\delta\theta$ is smaller and smaller. This is balanced by two forces each $q \,.\, \delta r$ inclined at the angle $\delta\theta$, and just as in page 165 if we draw a triangle, each of whose sides CA and AB is parallel to $q \,.\, \delta r$, the angle BAC being $\delta\theta$, and BC representing the radial force, we see that this radial force is $q \,.\, \delta r \,.\, \delta\theta$, and this expression is more and more nearly true as $\delta\theta$ is smaller and smaller. Hence

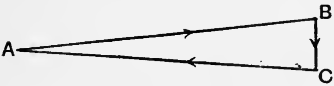

Fig. 35.

$$(p + \delta p)(r + \delta r)\,\delta\theta - pr \,.\, \delta\theta = q \,.\, \delta r \,.\, \delta\theta,$$

or $$p\,.\,\delta r + r\,.\,\delta p + \delta p\,.\,\delta r = q\,.\,\delta r,$$

or rather $$p + r\frac{dp}{dr} = q \quad\text{.......................(1)},$$

since the term δp is 0 in the limit.

When material is subjected to crushing stresses p and q in two directions at right angles to one another in the plane of the paper, the dimensions at right angles to the paper elongate by an amount which is proportional to $p + q$.

We must imagine the elongation to be independent of r if a plane cross section is to remain a plane cross section, and this reasonable assumption we make. Hence (1) has to be combined with $$p + q = 2A \quad\text{.......................(2)},$$

where $2A$ is a constant.

Substituting the value of q from (2) in (1) we have

$$p + r\frac{dp}{dr} = 2A - p,$$

or $$\frac{dp}{dr} = \frac{2A}{r} - \frac{2p}{r}.$$

Now it will be found on trial that this is satisfied by

$$p = A + \frac{B}{r^2} \quad\text{.......................(3)},$$

and hence from (2), $$q = A - \frac{B}{r^2} \quad\text{.......................(4)}.$$

To find these constants A and B. In the case of a gun or hydraulic press, subjected to pressure p_0 inside where $r = r_0$ and pressure 0 outside where $r = r_1$. Inserting these values of p in (3) we have

$$p_0 = A + \frac{B}{r_0^2},$$

$$0 = A + \frac{B}{r_1^2};$$

so that $$B = p_0 \div \left(\frac{1}{r_0^2} - \frac{1}{r_1^2}\right) = p_0\frac{r_0^2 r_1^2}{r_1^2 - r_0^2},$$

$$A = -p_0\frac{r_0^2}{r_1^2 - r_0^2};$$

and $$-q=\frac{p_0 r_0^2}{r_1^2-r_0^2}\left(1+\frac{r_1^2}{r^2}\right).$$

The compressive stress $-q$ may be called a tensile stress f.

$$f=p_0\frac{r_0^2}{r_1^2-r_0^2}\cdot\frac{r_1^2+r^2}{r^2} \quad\text{.....................(5)},$$

f is greatest at $r=r_0$ and is then

$$f_0=p_0\frac{r_1^2+r_0^2}{r_1^2-r_0^2} \quad\text{.........................(6)}.$$

This is the law of strength for a cylinder which is initially unstrained. Note that p_0 can never be equal to the tensile strength of the material. We see from (5) that as r increases, f diminishes in proportion to the inverse square of the radius, so that it is easy to show its value in a curve. Thus a student ought to take $r_1=1{\cdot}2$, $r_0=0{\cdot}8$, $p_0=1500$ lb. per sq. inch, and graph f from inside to outside. f will be in the same units as p. (5) may be taken as giving the tensile stress in a thick cylinder to resist bursting pressure if it is initially unstrained. If when $p_0=0$ there are already strains in the material, the strains produced by (5) are algebraically added to those already existing at any place. Hence in casting a hydraulic press we chill it internally, and in making a gun, we build it of tubes, each of which squeezes those inside it, and we try to produce such initial compressive strain at $r=r_0$ and such initial tensile strain at $r=r_1$, that when the tensile strains due to p_0 come on the material and the cylinder is about to burst there shall be much the same strain in the material from r_0 to r_1.*

* In the case of a cylindric body rotating with angular velocity a, if ρ is the mass per unit volume; taking into account the centrifugal force on the element whose equilibrium is considered, above the equation (1) becomes $p+r\frac{dp}{dr}-r^2\rho a^2=q$ and the solution of this is found to be $p=A+Br^{-2}+\frac{1}{4}\rho a^2 r^2$ and by inserting the values of p for two values of r we find the constants A and B; q is therefore known. If we take $p=0$ when $r=r_0$ and also when $r=r_1$,

$$q=\tfrac{1}{4}\rho a^2\left[-r^2+\{r_0^{-2}r_1^2-r_1^{-2}r_0^2+r^{-2}(r_1^2-r_0^2)\}/(r_1^{-2}-r_0^{-2})\right].$$

This is greatest when $r=r_0$.

If the cylinder extends to its centre we must write out the condition that the displacement is 0 where $r=0$, and it is necessary to write out the values

Thin cylinder. Take $r_0 = R$ and $r_1 = R + t$, where t is very small compared with R,

$$f_0 = p_0 \frac{2R^2 + 2Rt + t^2}{2Rt + t^2} = \frac{p_0 R}{t}\left(1 + \frac{t}{R} + \frac{t^2}{2R^2}\right) \Big/ \left(1 + \frac{t}{2R}\right).$$

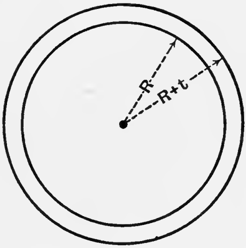

Fig. 36.

Now $\frac{t}{R}$ and $\frac{t^2}{2R^2}$ and $\frac{t}{2R}$ become all smaller and smaller as t is thought to be smaller and smaller. We may take (7) as a formula to be used when the shell is exceedingly thin and (8) as a closer approximation, which is the same as if we used the *average* radius in (7). In actual boiler and pipe work, there is so much uncertainty as to the proper value of f for ultimate strength, that we may neglect the correction of the usual formula (7).

$$f = \frac{pR}{t} \qquad \text{(7)},$$

$$f = \frac{pR}{t} + \frac{p}{2} \qquad \text{(8)}.$$

57. Gas Engine Indicator Diagram. It can be proved that when a perfect gas (whose law is $pv = Rt$ for a pound of gas, R being a constant and equal to $K - k$ the difference of the important specific heats; γ is used to denote K/k) changes in its volume and pressure in any

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of the strains. Radial strain $= p\alpha - q\beta$ if α is the reciprocal of Young's Modulus and β/α is Poisson's ratio, generally of the value 0·25.

In this way we find the strains and stresses in a rotating solid cylinder, but on applying our results to the case of a thin disc we see that equation (2) above is not correct. That is, the solution is less and less correct as the disc is thinner. Dr Chree's more correct solution is not difficult.

way, the rate of reception of heat by it per unit change of volume, which we call h (in work units) or $\frac{dH}{dv}$, is

$$h = \frac{1}{\gamma - 1}\frac{d}{dv}(pv) + p \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

or

$$h = \frac{1}{\gamma - 1}\left\{v\frac{dp}{dv} + \gamma p\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Students ought to note that this $\frac{dp}{dv}$ is a very different thing from $\left(\frac{dp}{dv}\right)$, because we may give to it any value we please.

We always assume Heat to be expressed in work units so as to avoid the unnecessary introduction of J for Joule's equivalent.

Exercise 1. When gas expands according to the law $pv^s = c$...(3) a constant, find h.

$$\text{Answer: } h = \frac{\gamma - s}{\gamma - 1}p \quad \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Evidently when $s = \gamma$, $h = 0$, and hence we have $pv^\gamma =$ constant as the adiabatic law of expansion of a perfect gas. γ is 1·41 for air and 1·37 for the stuff inside a gas or oil engine cylinder. When $s = 1$, so that the law of expansion is pv constant, we have the isothermal expansion of a gas, and we notice that here $h = p$, or the rate of reception of heat energy is equal to the rate of the doing of mechanical energy. Notice that in any case where the law of change is given by (3), h is exactly proportional to p. If s is greater than γ the stuff is having heat withdrawn from it.

If the equation (1) be integrated with regard to v we have $H_{01} = \frac{1}{\gamma - 1}(p_1 v_1 - p_0 v_0) + W_{01}$...(5). Here H_{01} is the heat given to a pound of perfect gas between the states p_0, v_0, t_0 and p_1, v_1, t_1, and W_{01} is the work done by it in expanding from the first to the second state.

This expression may be put in other forms because we have the connection $pv = Rt$...(6). It is very useful in calculations upon gas engines. Thus, if the volume keeps

constant W_{01} is 0 and the change of pressure due to ignition and the gift of a known amount of heat may be found. If the pressure keeps constant, W_{01} is $p(v_1 - v_0)$ and the change of volume due to the reception of heat is easily found.

Another useful expression is $\frac{dH}{dv} = k\frac{dt}{dv} + p \ldots (7)$ where k is a constant, being the specific heat at constant volume. Integrating this with regard to v we find

$$H_{01} = k(t_1 - t_0) + W_{01} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

This gives us exactly the same answer as the last method, and may at once be derived from (5) by (6). In this form one sees that if no work is done, the heat given is $k(t_1 - t_0)$ and also that if there is no change of temperature the heat given is equal to the work done.

58. Elasticity is defined as increase of stress ÷ increase of strain. Thus, Young's modulus of elasticity is tensile or compressive stress (or load per unit of cross section of a tie bar or strut) divided by the strain or fractional change of length. Modulus of rigidity or shearing elasticity is shear stress divided by shear strain. Volumetric elasticity e is fluid stress or increase of pressure divided by the fractional diminution of volume produced. Thus if fluid at p and v, changes to $p + \delta p$, $v + \delta v$: then the volumetric stress is δp and the volumetric compressive strain is $-\delta v/v$, so that by definition $e = -\delta p \div \frac{\delta v}{v}$, or $e = -v\frac{\delta p}{\delta v}$. The definition really assumes that the stress and strain are smaller and smaller without limit and hence $e = -v\frac{dp}{dv} \ldots (1)$.

Now observe that this may have any value whatsoever. Thus the elasticity at constant pressure is 0. The elasticity at constant volume is $-\infty$. To find the elasticity at constant temperature, we must find $\left(\frac{dp}{dv}\right)$: see Art. 30. As $pv = Rt$, $p = Rtv^{-1}$. Here Rt is to be constant, so that

$$\left(\frac{dp}{dv}\right) = -Rtv^{-2} \text{ and } e = Rtv^{-1} = p.$$

It is convenient to write this e_t and we see that K_t, the elasticity at constant temperature, is p. This was the value of the elasticity taken by Newton; by using it in his calculation of the velocity of sound he obtained an answer which was very different from the experimentally determined velocity of sound, because the temperature does not remain constant during quick changes of pressure.

Exercise. Find the elasticity of a perfect gas when the gas follows the law $pv^\gamma = c$, some constant. This is the adiabatic law which we found Art. 57, the law connecting p and v when there is no time for the stuff to lose or gain heat by conduction. $p = cv^{-\gamma}$, so that $\frac{dp}{dv} = -\gamma cv^{-\gamma-1}$, and

$$e = +v\gamma cv^{-\gamma-1} \text{ or } \gamma cv^{-\gamma}, \text{ or } \gamma p.$$

It is convenient to write this e_H, and we see that in a perfect gas $e_H = \gamma e_t$. When this value of the elasticity of air is taken in Newton's calculation, the answer agrees with the experimentally found velocity of sound.

59. Friction at a Flat Pivot. If we have a pivot of radius R carrying a load W and the load is uniformly distributed over the surface, the load per unit area is $w = W \div \pi R^2$. Let the angular velocity be α radians per second. On a ring of area between the radii r and $r + \delta r$ the load is $w 2\pi r \,.\, \delta r$, and the friction is $\mu w 2\pi r \,.\, \delta r$, where μ is the coefficient of friction. The velocity is $v = \alpha r$, so that the work wasted per second in overcoming friction at this elementary area is $\mu 2\pi w \alpha r^2 \,.\, \delta r$, so that the total energy wasted per second is

$$2\pi w \alpha \mu \int_0^R r^2 \,.\, dr = \tfrac{2}{3}\pi w \alpha \mu R^3 = \tfrac{2}{3}\alpha\,\iota W R.$$

On a collar of internal radius R_1 and external R_2 we have $2\pi w \alpha \mu \int_{R_1}^{R_2} r^2 \,.\, dr = \tfrac{2}{3}\pi w \alpha \mu (R_2^3 - R_1^3)$, $W = \pi w (R_2^2 - R_1^2)$ and hence, the energy wasted per second is $\tfrac{2}{3}\alpha\mu W \frac{R_2^3 - R_1^3}{R_2^2 - R_1^2}$.

60. Exercises in the Bending of Beams. When the Bending moment M at a section of a beam is known, we can calculate the curvature there, if the beam was

straight when unloaded, or the change of curvature if the unloaded beam was originally curved. This is usually written

$$\frac{1}{r} \text{ or } \frac{1}{r} - \frac{1}{r_0} = \frac{M}{EI},$$

where I is the moment of inertia of the cross section about a line through its centre of gravity, perpendicular to the plane of bending, and E is Young's modulus for the material. Thus, if the beam has a rectangular section of breadth b and depth d, then $I = \frac{1}{12}bd^3$ (see Art. 54); if the beam is circular in section, $I = \frac{\pi}{4}R^4$, if R is the radius of the section (see Art. 50). If the beam is elliptic in section, $I = \frac{\pi}{4}a^3b$, if a and b are the radii of the section in and at right angles to the plane of bending (see Art. 50). †

Curvature. The curvature of a circle is the reciprocal of its radius, and of any curve it is the curvature of the circle which best agrees with the curve. The curvature of a curve is also "the angular change (in radians) of the direction of the curve per unit length." Now draw a very flat curve, with very little slope. Observe that the change in $\frac{dy}{dx}$ in going from a point P to a point Q is almost exactly a change of angle [change in $\frac{dy}{dx}$ is really a change in the tangent of an angle, but when an angle is very small, the angle, its sine and its tangent are all equal]. Hence, the increase in $\frac{dy}{dx}$ from P to Q divided by the length of the curve PQ is the average curvature from P to Q, and as PQ is less and less we get more and more nearly the curvature at P. But the curve being very flat, the length of the arc PQ is really δx, and the change in $\frac{dy}{dx}$ divided by δx, as δx gets less and less, is the rate of change of $\frac{dy}{dx}$ with regard to x, and the symbol for this is $\frac{d^2y}{dx^2}$. Hence we may take $\frac{d^2y}{dx^2}$ as the curvature of a curve at any place, when its slope is everywhere small.

If the beam was not straight originally and if y' was its small deflection from straightness at any point, then $\frac{d^2y'}{dx^2}$ was its original curvature. We may generalize the following work by using $\frac{d^2}{dx^2}(y-y')$ instead of $\frac{d^2y}{dx^2}$ everywhere.

It is easy to show, that a beam of uniform strength, that is a beam in which the maximum stress f (if compressive; positive, if tensile, negative), in every section is the same, has the same curvature everywhere if its depth is constant.

If d is the depth, the condition for constant strength is that $\frac{M}{I}\cdot\frac{1}{2}d=\pm f$ a constant. But $\frac{M}{I}=E\times$ curvature, hence curvature $=\pm\frac{2f}{E\,.\,d}$.

Exercise. In a beam of constant strength if $d=\frac{1}{a+bx}$.

Then $\frac{d^2y}{dx^2}=\frac{2f}{E}(a+bx)$. Integrating we find

$$\frac{E}{2f}\cdot\frac{dy}{dx}=c+ax+\tfrac{1}{2}bx^2,\quad \frac{E}{2f}\cdot y=e+cx+\tfrac{1}{2}ax^2+\tfrac{1}{6}\,.\,bx^3,$$

where e and c must be determined by some given condition. Thus if the beam is fixed at the end, where $x=0$, and $\frac{dy}{dx}=0$ there, and also $y=0$ there, then $c=0$ and $e=0$.

In a beam originally straight we know now that, if x is distance measured from any place along the beam to a section, and if y is the deflection of the beam at the section, and I is the moment of inertia of the section, then

$$\mathbf{\frac{d^2y}{dx^2}=\frac{M}{EI}} \quad\text{.........................(1)},$$

where M is the bending moment at the section, and E is Young's modulus for the material.

We give to $\frac{d^2y}{dx^2}$ the sign which will make it positive if M is positive. If M would make a beam convex upwards and y is measured downwards then (1) is correct. Again, (1) would be right if M would make a beam concave upwards and y is measured upwards.

Example I. **Uniform beam of length l fixed at one end,** loaded with weight W at the other. Let x be the distance of a section from the fixed end of the beam. Then $M = W(l-x)$, so that (1) becomes

$$\frac{EI}{W}\frac{d^2y}{dx^2} = l - x \quad \text{.......................(2).}$$

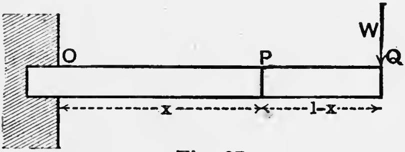

Fig. 37.

Integrating, we have, as E and I are constants,

$$\frac{EI}{W}\frac{dy}{dx} = lx - \tfrac{1}{2}x^2 + c.$$

From this we can calculate the slope everywhere.

To find c, we must know the slope at some one place. Now we know that there is no slope at the fixed end, and hence $\frac{dy}{dx} = 0$ where $x = 0$, hence $c = 0$. Integrating again,

$$\frac{EI}{W}y = \tfrac{1}{2}lx^2 - \tfrac{1}{6}x^3 + C.$$

To find C, we know that $y = 0$ when $x = 0$, and hence $C = 0$, so that we have for the shape of the beam, that is, the equation giving us y for any point of the beam,

$$y = \frac{W}{EI}\left(\tfrac{1}{2}lx^2 - \tfrac{1}{6}x^3\right) \quad \text{...................(3).}$$

We usually want to know y when $x = l$, and this value of y is called D, the deflection of the beam, so that

$$D = \frac{Wl^3}{3EI} \quad \text{.........................(4).}$$

Example II. A beam of length l **loaded** with W **at the middle and supported at the ends.** Observe that if half of this beam in its loaded condition has a casting of

cement made round it so that it is rigidly held; the other half is simply a beam of length $\frac{1}{2}l$, fixed at one end and

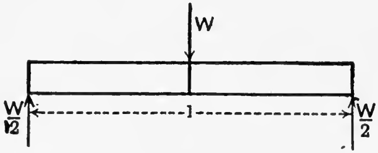

Fig. 38.

loaded at the other with $\frac{1}{2}W$, and, according to the last example, its deflection is

$$D = \frac{\frac{1}{2}W\left(\frac{1}{2}l\right)^3}{3EI} \text{ or } \frac{Wl^3}{48 \,.\, EI} \quad \text{.................(5).}$$

The student ought to make a sketch to illustrate this method of solving the problem.

Example III. Beam **fixed at one end** with load w per unit length **spread over it uniformly**.

The load on the part PQ is $w \times PQ$ or $w(l-x)$.

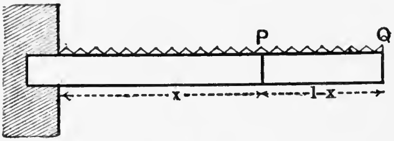

Fig. 39.

The resultant of the load acts at midway between P and Q, so, multiplying by $\frac{1}{2}(l-x)$, we find M at P, or

$$M = \frac{1}{2}w(l-x)^2 \quad \text{......................(6).}$$

Using this in (1), we have

$$\frac{2EI}{w}\frac{d^2y}{dx^2} = l^2 - 2lx + x^2.$$

Integrating, we have

$$\frac{2EI}{w}\frac{dy}{dx} = l^2x - lx^2 + \frac{1}{3}x^3 + c.$$

This gives us the slope everywhere.

Now $\frac{dy}{dx} = 0$ where $x = 0$, because the beam is fixed there. Hence $c = 0$.

Again integrating,

$$\frac{2EI}{w} y = \tfrac{1}{2}l^2x^2 - \tfrac{1}{3}lx^3 + \tfrac{1}{12}x^4 + C,$$

and as $y = 0$ where $x = 0$, $C = 0$, and hence the shape of the beam is

$$y = \frac{w}{24EI}(6l^2x^2 - 4lx^3 + x^4) \text{ ....(7)},$$

y is greatest at the end where $x = l$, so that the deflection is

$$D = \frac{w}{24EI} 3l^4 \text{ or } D = \frac{1}{8}\frac{Wl^3}{EI} \text{ ....(8)},$$

if $W = wl$, the whole load on the beam.

Example IV. Beam of length l **loaded uniformly** with w per unit length, **supported at the ends.**

Each of the supporting forces is half the total load. The moment about P of $\frac{1}{2}wl$, at the distance PQ, is against the hands of a watch, and I call this direction positive; the moment of the load $w(\frac{1}{2}l - x)$ at the average distance $\frac{1}{2}PQ$ is therefore negative, and hence the bending moment at P is

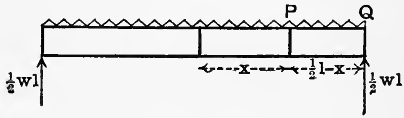

Fig. 40.

$$\tfrac{1}{2}wl(\tfrac{1}{2}l - x) - \tfrac{1}{2}w(\tfrac{1}{2}l - x)^2, \text{ or } \tfrac{1}{8}wl^2 - \tfrac{1}{2}wx^2 \text{ ...(9)},$$

so that, from (1), $EI\frac{d^2y}{dx^2} = \frac{1}{8}wl^2 - \frac{1}{2}wx^2$,

y being the vertical height of the point P *above* the middle of the beam, see Art. 60. Integrating we have

$$EI\frac{dy}{dx} = \tfrac{1}{8}wl^2x - \tfrac{1}{6}wx^3 + c,$$

a formula which enables us to find the slope everywhere.

c is determined by our knowledge that $\frac{dy}{dx}=0$ where $x=0$, and hence $c=0$. Integrating again,

$$EIy=\tfrac{1}{16}wl^2x^2-\tfrac{1}{24}wx^4+C,$$

and $C=0$, because $y=0$, where $x=0$. Hence the shape of the beam is $y=\frac{w}{48EI}(3l^2x^2-2x^4)\ldots(10)$, y is greatest where $x=\frac{1}{2}l$, and is what is usually called the deflection D of the beam, or $D=\frac{5Wl^3}{384EI}$ if $W=lw$ the total load.

61. Beams Fixed at the Ends. Torques applied at the ends of a beam to fix them (that is, to keep the end sections in vertical planes) are equal and opposite if the loading is symmetrical on the two sides of the centre of the beam. The torques being equal, the supporting forces are the same as before. Now if m is the bending moment (positive if the beam tends to get concave upwards) which the loads and supporting forces would produce *if the ends were not fixed*, the bending moment is now $m-c$ because the end torques c are equal and opposite, and the supporting forces are unaltered by fixing.

Thus $$\frac{d^2y}{dx^2}=\frac{m-c}{EI}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

If the beam is uniform and we integrate, we find

$$EI\,.\,\frac{dy}{dx}=\int m\,.\,dx-cx+\text{const.}\ \ldots\ldots\ldots\ldots(2).$$

Take x as measured from one end. We have the two conditions: $\frac{dy}{dx}=0$ where $x=0$, and $\frac{dy}{dx}=0$ where $x=l$, if l is the length of the beam. Hence if we subtract the value of (2) when $x=0$ from what it is when $x=l$, we have

$$0=\int_0^l m\,.\,dx-cl,\ \text{ or }\ c=\frac{1}{l}\int_0^l m\,.\,dx,$$

that is, **c is the average value of m all over the beam.** The rule is then (for symmetric loads):—Draw the diagram of bending moment m as if the beam were merely *supported*

at the ends. Find the average height of the diagram and *lower* the curved outline of the diagram by that amount. The resulting diagram, which will be negative at the ends, is the true diagram of bending moment. The beam is concave upwards where the bending moment is positive, and it is convex upwards where the bending moment is negative, and there are points of inflexion, or places of no curvature, where there is no bending moment.

Example. Thus it is well known that if a beam of length l is **supported at the ends and loaded in the middle** with a load W, the bending moment is $\frac{1}{4}Wl$ at the middle and is 0 at the ends, the diagram being formed of two straight lines. The student is supposed to draw this diagram (see also *Example II.*). The average height of it is half the middle height or $\frac{1}{8}Wl$, and this is c the torque which must be applied at each end to fix it **if the ends are fixed.** The whole diagram being lowered by this amount it is evident that the true bending moment of such a beam if its ends are fixed, is $\frac{1}{8}Wl$ at the middle, 0 half-way to each end from the middle so that there are points of inflexion there, and $-\frac{1}{8}Wl$ at each end. A rectangular beam or a beam of rolled girder section, or any other section symmetrical above and below the neutral line, is equally ready to break at the ends or at the middle.

Example. **A uniform beam loaded uniformly** with load w per unit length, **supported at the ends;** the diagram for m is a parabola (see *Example IV.*, where $M = \frac{1}{8}wl^2 - \frac{1}{2}wx^2$); the greatest value of m is at the middle and it is $\frac{1}{8}wl^2$; m is 0 at the ends. Now the average value of m is $\frac{2}{3}$ of its middle value (see Art. 43, area of a parabola). Hence $c = \frac{1}{12}wl^2$. This average value of m is to be subtracted from every value and we have the value of the real bending moment everywhere **for a beam fixed at the ends.**

Hence in such a beam fixed at the ends the bending moment in the middle is $\frac{1}{24}wl^2$, at the ends $-\frac{1}{12}wl^2$, and the diagram is parabolic, being in fact the diagram for a beam supported at the ends, lowered by the amount $\frac{1}{12}wl^2$ everywhere. The points of inflexion are nearer the ends than in the last case. The beam is most likely to break at the ends.

Students ought to make diagrams for various examples of

symmetrical loading. Find m by the ordinary graphical method and lower the diagram by its average height.

When the beam symmetrically loaded and fixed at the ends is not uniform in section, the integral of (1) is

$$E\frac{dy}{dx}=\int\frac{m}{I}\,dx-c\int\frac{dx}{I}\;\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

and as before this is 0 between the limits 0 and l, and hence to find c it is necessary to draw a diagram showing the value of $\frac{m}{I}$ everywhere and to find its area. Divide this by the area of a diagram which shows the value of $\frac{1}{I}$ everywhere, or the average height of the M/I diagram is to be divided by the average height of the $1/I$ diagram and we have c. Subtract this value of c from every value of m, and we have the true diagram of bending moment of the beam. Graphical exercises are much more varied and interesting than algebraic ones, as it is so easy, graphically, to draw diagrams of m when the loading is known.

The solution just given is applicable to a beam of which the I of every cross section is settled beforehand in any arbitrary manner, so long as I and the loading are symmetrical on the two sides of the middle. Let us give to I such a value that the beam shall be of **uniform strength** everywhere; that is, that $\frac{M}{I}z=f_c$ or f_t...(4), where z is the greatest distance of any point in the section from the neutral line on the compression or tension side and f_c and f_t are the constant maximum stresses in compression or tension to which the material is subjected in every section. Taking f_c as numerically equal to f_t and $z=\frac{1}{2}d$, where d is the depth of the beam, (4) becomes $\frac{M}{I}d=\pm 2f$...(5), the $+$ sign being taken over parts of the beam where M is positive, the $-$ sign where M is negative. As $\int_0^l\frac{M}{I}\,dx=0$, or, using (5),

$$\int\pm\frac{2f}{d}\,dx=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

the negative sign being taken from the ends of the beam to the points of inflexion, and the positive sign being taken between the two points of inflexion. We see then that to satisfy (6) we have only to solve the following problem. In the figure, *EATUCGE* is a diagram whose ordinates represent the values of $\frac{1}{d}$ or the reciprocal of the depth of the beam which may be arbitrarily fixed, care being taken, however, that d is the same at points which are at the same distance from the centre. *EFGE* is a diagram of the values of m easily drawn when the loading is known. We are required to find a point P, such that the area of $EPTA =$ area

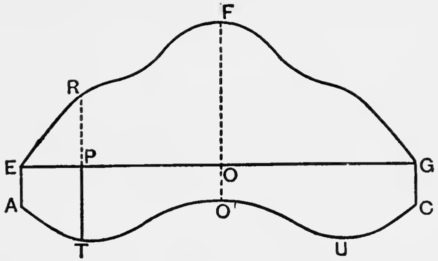

Fig. 41.

of $POO'T$, where O is in the middle of the beam. When found, this point P is a point of inflexion and PR is what we have called c. That is, $m - PR$ is the real bending moment M at every place, or the diagram *EFG* must be lowered vertically till R is at P to obtain the diagram of M. Knowing M and d it is easy to find I through (5).

It is evident that if such a beam of **uniform strength** is also of **uniform depth,** the points of inflexion are half-way between the middle and the fixed **ends.** Beams of uniform strength and depth are of the same curvature everywhere except that it suddenly changes sign at the points of inflexion.

61. In the **most general way of loading, the bending moments required at the ends to fix them are different from one another,** and if m_1 is the torque against the hands of a watch applied at the end A, and m_2 is the torque with the hands of a

watch at the end B, and if the bending moment in case the beam were merely supported is m:—

Consider a weightless unloaded beam of the same length with the torques m_1 and m_2 applied to its ends; to keep it in equilibrium it is necessary to introduce equal and opposite supporting forces P at the

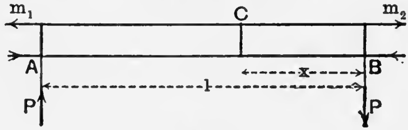

Fig. 42.

ends as shown in the figure. Then $Pl + m_2 = m_1$, the forces &c. being as drawn in fig. 42, so that $P = \frac{m_1 - m_2}{l}$.

If then these torques m_2 and m_1 are exerted they must be balanced by the forces P shown; that is, at B a downward force must be exerted; this means that the beam at B tends to rise, and hence the ordinary supporting force at B must be *diminished* by amount P. At any place C the bending moment will be m (what it would be if the beam were merely supported at the ends) $- m_2 - P \,.\, BC$...(1). If one does not care to think much, it is sufficient to say:—The beam was in equilibrium being loaded and merely supported at the ends; the bending moment at any place was m; we have introduced now a new set of forces which balance, the bending moment at C due to these new forces is

$$-(m_2 + P \,.\, BC).$$

So that the true bending moment at C is $m - m_2 - P \,.\, BC$.

Suppose $m_2 = 0$, then $P = \frac{m_1}{l}$, and the bending moment at C is $m - \frac{m_1}{l} \,.\, BC$ or $m - P \,.\, BC$.

62. **Beam fixed at the end A, merely supported at B** which is exactly on the same level as A. As $m_2 = 0$ and letting $BC = x$, we have the very case just mentioned, and

$$EI \frac{d^2y}{dx^2} = m - Px \text{ .... } (2).$$

We will first consider a **uniform beam uniformly loaded** as in Example IV., Art. 60. It will be found that when x is measured from the end of the beam, the bending moment $m = \frac{1}{2} wlx - \frac{1}{2} wx^2$, if the beam is merely supported at its ends and w is the load per unit length. Hence (2) is

$$EI \frac{d^2y}{dx^2} = \tfrac{1}{2} wlx - \tfrac{1}{2} wx^2 - Px \text{ .... } (3),$$

$$EI \frac{dy}{dx} = \tfrac{1}{4} wlx^2 - \tfrac{1}{6} wx^3 - \tfrac{1}{2} Px^2 + c \text{ .... } (4).$$

We have also the condition that $\frac{dy}{dx}=0$ where $x=l$...(5), for it is to be observed that we measure x from the unfixed end.

Again integrating,

$$EIy=\tfrac{1}{12}wlx^3-\tfrac{1}{24}wx^4-\tfrac{1}{6}Px^3+cx \quad\text{.................(6).}$$

We need not add a constant because y is 0 when x is 0.

We also have $y=0$ when $x=l$. Using this condition and also (5) we find

$$0=\tfrac{1}{12}wl^3-\tfrac{1}{2}Pl^2+c \quad\text{..........................(7),}$$

$$0=\tfrac{1}{24}wl^4-\tfrac{1}{6}Pl^3+cl \quad\text{.......................(8),}$$

and these enable us to determine P and c.

Divide (8) by l and subtract from (7) and we have

$$0=\tfrac{1}{24}wl^3-\tfrac{1}{3}Pl^2 \text{ or } P=\tfrac{1}{8}wl,$$

hence from (7), $\quad 0=\tfrac{1}{12}wl^3-\tfrac{1}{16}wl^3+c,\ c=-\tfrac{1}{48}wl^3.$

We have the true bending moment,

$$\tfrac{1}{2}wlx-\tfrac{1}{2}wx^2-\tfrac{1}{8}wlx,$$

and (6) gives us the shape of the beam.

63. **If the loading is of any kind whatsoever and if the section varies in any way** a graphic method of integration must be used in working the above example. Now if the value of an ordinate z which is a function of x be shown on a curve, we have no instrument which can be relied upon for showing in a new curve $\int z\,.\,dx$, that is, the ordinate of the new curve representing the area of the z curve up to that value of x from any fixed ordinate. I have sometimes used squared paper and counted the number of the squares. I have sometimes used a planimeter to find the areas up to certain values of x, raised ordinates at those places representing the areas to scale, and drawn a curve by hand through the ten or twelve or more points so found. There are integrators to be bought; I have not cared to use any of them, and perhaps it is hardly fair to say that I do not believe in the accuracy of such of them as I have seen.

A cheap and accurate form of integrator would not only be very useful in the solution of graphical problems; it would, if it were used, give great aid in enabling men to understand the calculus.

Let us suppose that the student has some method of showing the value of $\int_0^x z\,.\,dx$ in a new curve; the loading being of any kind whatsoever and I varying, since

$$M=m-Px=EI\frac{d^2y}{dx^2},$$

we have on integrating,

$$E\frac{dy}{dx}=\int\frac{m}{I}\,dx-P\int\frac{x\,.\,dx}{I}-c\,\text{................}\,(9).$$

We see that it is necessary to make a diagram whose ordinate everywhere is $\frac{m}{I}$ and we must integrate it. Let $\int_0^x\frac{m}{I}\,dx$ be called μ; when $x=l$, μ becomes the whole area of the $\frac{m}{I}$ diagram and we will call this μ_1.

It is also necessary to make a diagram whose ordinate everywhere is $\frac{x}{I}$ and integrate it. Let $\int_0^x\frac{x}{I}\,dx$ be called X. When $x=l$, X becomes the whole area of the $\frac{x}{I}$ diagram and we will call this X_1.

Then as in (9), $\frac{dy}{dx}=0$, when $x=l$,

$$0=\mu_1-P\,.\,X_1+c\,\text{........................}\,(10).$$

Integrating (9) again, we have

$$Ey=\int\mu\,.\,dx-P\int X\,.\,dx+cx+C.$$

In this if we use $y=0$ when $x=0$, we shall find $C=0$, and again if $y=0$ when $x=l$, and if we use $\mathbf{M}_1$ and $\mathbf{X}_1$ as the total areas of the μ and X curves we have

$$0=\mathbf{M}_1-P\,.\,\mathbf{X}_1+cl^*\,\text{......................}\,(11),$$

from (10) and (11) P and c may be found, and of course P enables us to state the bending moment everywhere. $-c$ is the slope when x is 0.

64. *Example.* **Beam of any changing section fixed**

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* Without using the letters μ, X, μ_1, X_1 &c. the above investigation is:—

$$\left[E\frac{dy}{dx}\right]_{x=0}^{x=l}=-c=\int_0^l\frac{m}{I}\,dx-P\int_0^l\frac{x\,.\,dx}{I}\,\text{.................}\,(10).$$

Integrating again between the limits 0 and l and recollecting that y is the same at both limits

$$\left[Ey\right]_{x=0}^{x=l}=0=\int_0^l\int_0^l\frac{m}{I}\,dx-P\int_0^l\int_0^l\frac{x\,.\,dx}{I}+cl\,\text{...........}\,(11).$$

The integrations in (10) and (11) being performed, the unknowns P and c can be calculated; the true bending moment everywhere is what we started with,

$$m-Px.$$

at the ends, any kind of loading. Measuring x from one end where there is the fixing couple m_2,

$$M = m - m_2 - Px \text{ .... } (1),$$

$$E\frac{d^2y}{dx^2} = \frac{m}{I} - \frac{m_2}{I} - P\frac{x}{I} \text{ .... } (2),$$

$$E\frac{dy}{dx} = \int\frac{m}{I}\,.\,dx - m_2\int\frac{dx}{I} - P\int\frac{x\,.\,dx}{I} + \text{constant ...... } (3),$$

Let $\mu = \int\frac{m\,.\,dx}{I}$ and μ_1 the whole area of the $\frac{m}{I}$ curve; Let $Y = \int\frac{dx}{I}$ and Y_1 the whole area of the $\frac{1}{I}$ curve; Let $X = \int\frac{x\,.\,dx}{I}$ and X_1 the whole area of the $\frac{x}{I}$ curve; then

$$0 = \mu_1 - m_2Y_1 - PX_1 \text{ .... } (4).$$

Again integrating

$$y = \int\mu\,.\,dx - m_2\int Y\,.\,dx - P\int X\,.\,dx + \text{const.}$$

Calling the integrals from 0 to l of the μ, Y and X curves $\mathbf{M}_1$, $\mathbf{Y}_1$, and $\mathbf{X}_1$, we have

$$0 = \mathbf{M}_1 - m_2\mathbf{Y}_1 - P\mathbf{X}_1{}^* \text{ .... } (5),$$

and as m_2 and P are easily found from (4) and (5), (1) is known.

---

* We have used the symbols μ, X, Y, μ_1, X_1, Y_1, **M**, **Y**, $\mathbf{M}_1$, $\mathbf{X}_1$, $\mathbf{Y}_1$ fearing that students are still a little unfamiliar with the symbols of the calculus; perhaps it would have been better to put the investigation in its proper form and to ask the student to make himself familiar with the usual symbol instead of dragging in eleven fresh symbols.

After (3) above, write as follows;—

$$\left[E\frac{dy}{dx}\right]_0^l = 0 = \int_0^l\frac{m}{I}dx - m_2\int_0^l\frac{dx}{I} - P\int_0^l\frac{x\,dx}{I} \text{ .... } (4).$$

Again integrating between limits

$$\left[y\right]_{x=0}^{x=l} = 0 = \int_0^l\int_0^l\frac{m}{I}dx - m_2\int_0^l\int_0^l\frac{dx}{I} - P\int_0^l\int_0^l\frac{x\,dx}{I} \text{ .... } (5).$$

The integrations indicated in (4) and (5) being performed, the unknowns m_2 and P can be calculated and used in (1). The student must settle for himself which is the better course to take; to use the formidable looking but really easily understood symbols of this note or to introduce the eleven letters whose meaning one is always forgetting. See also the previous note.

65. In Graphical work. Let ACB (fig. 43) represent m, the bending moment, if the beam were merely supported at the ends; let AD represent m_1 and let BE represent m_2. Join DE. Then the difference between the ordinates of

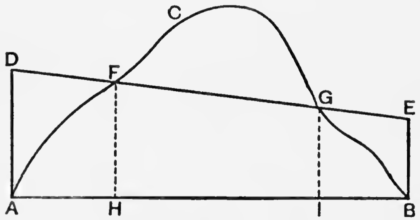

Fig. 43.

ACB and of $ADEB$ represents the actual bending moment; that is the vertical ordinates of the space between the straight line DE and the curve $AFCGB$. It is negative from A to H and from I to B, and positive from H to I. F and G are points of inflexion.

66. Useful Analogies in Beam Problems. If w is the load per unit length on a beam and M is the bending moment at a section (positive when it tends to make the beam convex upwards*), x being horizontal distance, to prove that

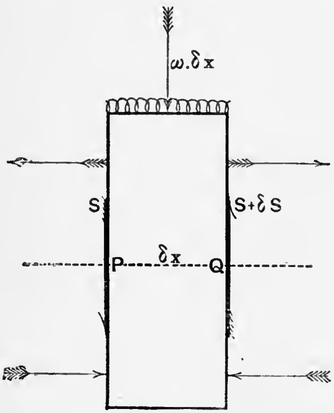

Fig 44.

$$\frac{\mathbf{d^2M}}{\mathbf{dx^2}} = \mathbf{w} \quad \ldots\ldots\ldots(1).$$

If at the section at P fig. 44, whose distance to the right of some origin is x there is a bending moment M indicated by the two equal and opposite arrow heads and a shearing force S as shown, being positive if the material to the right of the section is

* This convention is necessary only in the following generalization.

acted on by downward force, and if PQ is δx so that the load on this piece of beam between the sections at P and Q is $w . \delta x$; if the bending moment on the Q section is $M + \delta M$ and the shearing force $S + \delta S$, then the forces acting on this piece of beam are shown in the figure and from their equilibrium we know that

$$\delta S = w . \delta x \text{ or } \frac{\mathbf{dS}}{\mathbf{dx}} = \mathbf{w} \quad \ldots\ldots\ldots\ldots\ldots(2),$$

and taking moments about Q,

$$M + S . \delta x + \tfrac{1}{2} w (\delta x)^2 = M + \delta M,$$

or
$$\frac{\delta M}{\delta x} = S + \tfrac{1}{2} w . \delta x,$$

and in the limit as δx is made smaller and smaller

$$\frac{\mathbf{dM}}{\mathbf{dx}} = \mathbf{S} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

and hence (1) is true.

Now it is well known that in beams if y is the deflection

$$\frac{\mathbf{d^2y}}{\mathbf{dx^2}} = \frac{\mathbf{M}}{\mathbf{EI}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

If we have a diagram which shows at every place the value of w, called usually a diagram of loading, it is an exercise known to all students that we can draw at once by graphical statics a diagram showing the value of M at every place to scale; that is we can solve (1) very easily graphically*. We can see from (4) that if we get a diagram showing $\frac{M}{EI}$ at every place, we can use exactly the same method (and we have exactly the same rule as to scale) to find the value of y; that is, to draw the shape of the beam. Many of these exercises ought to be worked by all engineers.

* We find M usually for a beam merely supported at the ends. Let it be ACB, fig. 43. If instead, there are bending moments at the ends we let AD and BE represent these and join DE. Then the algebraic sum of the ordinates of the two diagrams is the real diagram of bending moment.

Example. In any beam whether supported at the ends or not: if w is constant, integrating (1) we find

$$\frac{dM}{dx} = b + wx \text{ and } M = a + bx + \tfrac{1}{2}wx^2 \quad \ldots\ldots(5).$$

In any problem we have data to determine a and b.

Take the case of a uniform beam uniformly loaded and merely supported at the ends.

Measure y upwards from the middle and x from the middle. Then $M = 0$ where $x = \frac{1}{2}l$ and $-\frac{1}{2}l$,

$$0 = a + \tfrac{1}{2}bl + \tfrac{1}{8}wl^2,$$

and

$$0 = a - \tfrac{1}{2}bl + \tfrac{1}{8}wl^2.$$

Hence $b = 0$, $a = -\frac{1}{8}wl^2$ and (5) becomes

$$M = -\tfrac{1}{8}wl^2 + \tfrac{1}{2}wx^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

which is exactly what we used in *Example IV.* (Art. 60) where we afterwards divided M by EI and integrated twice to find y.

Let i be $\dfrac{dy}{dx}$ or the slope of the beam.

Since $\quad \dfrac{dy}{dx} = i,\ \dfrac{di}{dx} = \dfrac{M}{EI},\ \dfrac{dM}{dx} = S,\ \dfrac{dS}{dx} = w,$

we have a succession of curves which may be obtained from knowing the shape of the beam y by differentiation, or which may be obtained from knowing w, the loading of the beam, by integration. Knowing w there is an easy graphical rule for finding M/EI, knowing M/EI we have the same graphical rule for finding y. Some rules that are obviously true in the w to M/EI construction and need no mathematical proof, may at once be used without mathematical proof in applying the analogous rule from M/EI to y. Thus the area of the M/EI curve between the ordinates x_1 and x_2 is the increase of i from x_1 to x_2, and tangents to the curve showing the shape of the beam at x_1 and x_2 meet at a point which is vertically in a line with the centre of gravity of the portion of area of the M/EI curve in question. Thus the whole area of the M/EI curve in a span HJ is equal to the increase in $\dfrac{dy}{dx}$ from one end of the span to the other, and

the tangents to the beam at its ends H, J meet in a point P which is in the same vertical as the centre of gravity of the whole M/EI curve. These two rules may be taken as the starting point for a complete treatment of the subject of beams by graphical methods.

If the vertical from this centre of gravity is at the horizontal distance HG from H and GJ from J, then P is higher than H by the amount $HG \times i_H$, the symbol i_H being used to mean the slope at H; J is higher than P by the amount $GJ \times i$ at J. Hence J is higher than H by the amount

$$HG \,.\, i_H + GJ \,.\, i_J,$$

a relation which may be useful when conditions as to the relative heights of the supports are given, as in continuous beam problems.

67. Theorem of Three Moments. For some time, Railway Engineers, instead of using separate girders for the spans of a bridge, fastened together contiguous ends to prevent their tilting up and so made use of what are called **continuous girders.** It is easy to show that if we can be absolutely certain of the positions of the points of support, continuous girders are much cheaper than separate girders. Unfortunately a comparatively small settlement of one of the supports alters completely the condition of things. In many other parts of Applied Mechanics we have the same difficulty in deciding between cheapness with some uncertainty and a greater expense with certainty. Thus there is much greater uncertainty as to the nature of the forces acting at riveted joints than at hinged joints and therefore a structure with hinged joints is preferred to the other, although, if we could be absolutely certain of our conditions an equally strong riveted structure might be made which would be much cheaper.

Students interested in the theory of continuous girders will do well to read a paper published in the *Proceedings of the Royal Society*, 199, 1879, where they will find a graphical method of solving the most general problems.† I will take here as a good example of the use of the calculus, a uniform girder resting on supports at the same level, with a uniform

load distribution on each span. Let ABC be the centre line of two spans, **the girder originally straight,** supported at A, B and C. The distance from A to B is l_1 and from B to C

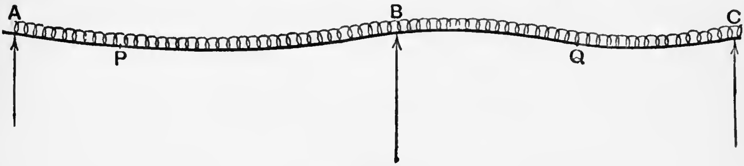

Fig. 45.

is l_2 and there are any kinds of loading in the two spans. Let A, B and C be the bending moments at A, B and C respectively, counted positive if the beam is *concave* upwards.

At the section at P at the distance x from A let m be what the bending moment would have been if the girder on each span were quite separate from the rest. We have already seen that by introducing couples m_2 and m_1 at A and B (tending to make the beam convex upwards at A and B) we made the bending moment at P really become what is given in Art. 61. Our $m_2 = -A$, $m_1 = -B$, and hence the bending moment at P is

$$m + A + x\frac{B-A}{l_1} = EI\frac{d^2y}{dx^2} \quad \text{...........(1)},$$

where m would be the bending moment if the beam were merely supported at the ends, and the supporting force at A is lessened by the amount

$$\frac{A-B}{l_1} \quad \text{.........................(2)}.$$

Assume EI constant and integrate with regard to x and we have

$$\int m\,.\,dx + Ax + \tfrac{1}{2}x^2\frac{B-A}{l_1} + c_1 = EI\,.\,\frac{dy}{dx} \quad \text{......(3)}.$$

Using the sign $\iint m\,.\,dx\,.\,dx$ to mean the integration of the curve representing $\int m\,.\,dx$ we have

$$\iint m\,.\,dx\,.\,dx + \tfrac{1}{2}Ax^2 + \tfrac{1}{6}x^3\frac{B-A}{l_1} + c_1x + e = EI\,.\,y \text{...(4)}.$$

As y is 0 when $x=0$ and it is evident that $\iint m.dx.dx=0$ when $x=0$, e is 0. Again $y=0$ when $x=l_1$. Using the symbol μ_1 to indicate the sum $\iint m.dx.dx$ over the whole span,

$$\mu_1+\tfrac{1}{2}Al_1^2+\tfrac{1}{6}l_1^2(B-A)+c_1l_1=0\;\ldots\ldots\ldots\ldots(5).$$

From (3) let us calculate the value of $EI\frac{dy}{dx}$ at the point B, and let us use the letter a_1 to mean the area of the m curve over the span, or $\int_0^{l_1} m.dx$, so that $EI\frac{dy}{dx}$ at B is

$$a_1+Al_1+\tfrac{1}{2}l_1(B-A)+c_1\;\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

But at any point Q of the second span, if we had let $BQ=x$ we should have had the same equations as (1), (3) and (4) using the letters B for A and C for B and the constant c_2. Hence making this change in (3) and finding $EI\frac{dy}{dx}$ at the point B where $x=0$, we have (6) equal to c_2 or

$$c_2-c_1=a_1+Al_1+\tfrac{1}{2}l_1(B-A)\;\ldots\ldots\ldots\ldots(7),$$

and instead of (5) we have

$$\mu_2+\tfrac{1}{2}Bl_2^2+\tfrac{1}{6}l_2^2(C-B)+c_2l_2=0\;\ldots\ldots(8).$$

Subtracting (5) from (8) after dividing by l_1 and l_2 we have

$$c_2-c_1=\frac{\mu_1}{l_1}-\frac{\mu_2}{l_2}+\tfrac{1}{2}Al_1-\tfrac{1}{2}Bl_2+\tfrac{1}{6}l_1(B-A)-\tfrac{1}{6}l_2(C-B)\ldots(9).$$

The equality of (7) and (9) is

$$Al_1+2B(l_1+l_2)+Cl_2=6\left(\frac{\mu_1}{l_1}-a_1-\frac{\mu_2}{l_2}\right)\ldots(10),$$

an equation connecting A, B and C, the bending moments at three consecutive supports. If we have any number of supports and at the end ones we have the bending moments 0 because the girder is merely supported there, or if we have two conditions given which will enable us to find them in case the girder is fixed or partly fixed, note that by writing

down (10) for every three consecutive supports we have a sufficient number of equations to determine all the bending moments at the supports.

Example. Let the loads be w_1 and w_2 per unit length over two consecutive spans of lengths l_1 and l_2. Then

$$m = \tfrac{1}{2}wlx - \tfrac{1}{2}wx^2, \int m \, . \, dx = \tfrac{1}{4}wlx^2 - \tfrac{1}{6}wx^3,$$

Hence $\quad a_1 = \dfrac{w_1}{12} l_1^3$, and $\displaystyle\iint m \, . \, dx \, . \, dx = \tfrac{1}{12}wlx^3 - \tfrac{1}{24}wx^4.$

Hence $$\mu_1 = \tfrac{1}{24}w_1 l_1^4, \; \mu_2 = \tfrac{1}{24}w_2 l_2^4.$$

Hence $\quad \dfrac{\mu_2}{l_2} + a_1 - \dfrac{\mu_1}{l_1}$ becomes $\tfrac{1}{24}w_2 l_2^3 + \dfrac{w_1}{12} l_1^3 - \tfrac{1}{24}w_1 l_1^3,$

or $$\tfrac{1}{24}(w_2 l_2^3 + w_1 l_1^3),$$

and hence the theorem becomes in this case

$$Al_1 + 2B(l_1 + l_2) + Cl_2 + \tfrac{1}{4}(w_2 l_2^3 + w_1 l_1^3) = 0 \ldots\ldots(10).$$

If the spans are similar and similarly loaded then

$$A + 4B + C + \tfrac{1}{2}wl^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots(11).$$

Case 1. A uniform and uniformly loaded beam rests on three equidistant supports. Here $A = C = 0$ and $B = -\tfrac{1}{8}wl^2$. $m = \tfrac{1}{2}w(lx - x^2)$, and hence the bending moment at a point P distant x from A is

$$\tfrac{1}{2}w(lx - x^2) + 0 - \frac{x}{l}\tfrac{1}{8}wl^2.$$

The supporting force at A is lessened from what it would be if the part of the beam AB were distinct by the amount shewn in (2), $\dfrac{A - B}{l_1}$ or $\tfrac{1}{8}wl$. It would have been $\tfrac{1}{2}wl$, so now it is really $\tfrac{3}{8}wl$ at each of the end supports, and as the total load is $2wl$, there remains $\tfrac{10}{8}wl$ for the middle support.

Case 2. A uniform and uniformly loaded beam rests on four equidistant supports, and the bending moments at these supports are A, B, C, D. Now $A = D = 0$ and from symmetry $B = C$. Thus (11) gives us

$$0 + 5B + \tfrac{1}{2}wl^2 = 0 \text{ or } B = C = -\tfrac{1}{10}wl^2.$$

If the span AB had been distinct, the first support would have had the load $\frac{1}{2}wl$, it now has $\frac{1}{2}wl - \frac{1}{10}wl$ or $\frac{4}{10}wl$. The supporting force at D is also $\frac{4}{10}wl$. The other two supports divide between them the remainder of the total load which is altogether $3wl$ and so each receives $\frac{11}{10}wl$. The supporting forces are then $\frac{4}{10}wl$, $\frac{11}{10}wl$, $\frac{11}{10}wl$ and $\frac{4}{10}wl$.

68. Shear Stress in Beams. Let the distance measured from any section of a beam, say at O, fig. 46, to the section at A be x, and let $OB = x + \delta x$. Let the bending moment at $C'AC$ be M and at $D'BD$ be $M + \delta M$,

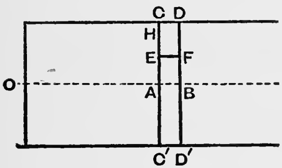

Fig. 46.

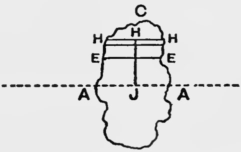

Fig. 47.

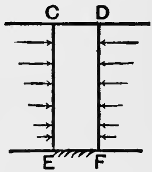

Fig. 48.

OAB (fig. 46) and AA (fig. 47) represent the neutral surface. We want to know the tangential or shear stress f at E on the plane CAC'. Now it is known that this is the same as the tangential stress in the direction EF on the plane EF which is at right angles to the paper and parallel to the neutral surface at AB. Consider the equilibrium of the piece of beam $ECDF$, shown in fig. 47 as ECE, and shown magnified in fig. 48. We have indicated only the forces which are parallel to the neutral surface or at right angles to the sections. *The total pushing forces on DF are greater than the total pushing forces on CE, the tangential forces on EF making up for the difference.* We have only to state this mathematically and we have solved our problem.

At a place like H in the plane CAC' at a distance y from the neutral surface the compressive stress is known to be $p = \frac{M}{I}y$, and if b is the breadth of the section there, shown as HH (fig. 47), the total pushing force on the area ECE is

$$P = \int_{AE}^{AC} b\frac{M}{I}y\,.\,dy \text{ or } P = \frac{M}{I}\int_{AE}^{AC} by\,.\,dy \ldots\ldots(1).$$

Observe that if b varies, we must know it as a function of y before we can integrate in (1). Suppose we call this total pushing force on EC by the name P, then the total pushing force on DF will be $P + \delta x \, . \, \frac{dP}{dx}$. The tangential force on EF is $f \times$ area of EF or $f \, . \, \delta x \, . \, EE$, and hence

$$f \, . \, \delta x \, . \, EE = \delta x \, . \, \frac{dP}{dx} \text{ or } f = \frac{1}{EE} \cdot \frac{dP}{dx} \text{ ....(2).}$$

Example. **Beam of uniform rectangular section,** of constant breadth b and constant depth d. Then

$$P = \frac{12Mb}{bd^3} \int_{AE}^{\frac{1}{2}d} y \, . \, dy = \frac{12M}{d^3} \left[\tfrac{1}{2} y^2 \right]_{AE}^{\frac{1}{2}d},$$

$$P = \frac{6M}{d^3} \left(\tfrac{1}{4} d^2 - AE^2 \right),$$

and hence
$$f = \frac{1}{b} \frac{6}{d^3} \left(\tfrac{1}{4} d^2 - AE^2 \right) \frac{dM}{dx} \text{ ....(3);}$$

so that f is known as soon as M is known.

As to M, let us choose a case, say the case of **a beam supported at the ends and loaded uniformly** with w lb. per unit length of the beam. We saw that in this case, x being distance from the middle

$$M = \tfrac{1}{8} w l^2 - \tfrac{1}{2} w x^2.$$

Hence $\frac{dM}{dx} = -wx$, so that (3) is

$$f = \frac{-6}{bd^3} \left(\tfrac{1}{4} d^2 - AE^2 \right) wx \text{ ....(4).}$$

If we like we may now use the letter y for the distance AE, and we see that at any point of this beam, x inches measured horizontally from the middle, and y inches above the neutral line the shear stress is

$$f = -\frac{6w}{bd^3} \left(\tfrac{1}{4} d^2 - y^2 \right) x \text{ ....(5).}$$

The $-$ sign means that the material below EF acts on the material above EF in the opposite sense to that of the arrow heads shown at EF, fig. 48.

Observe that where $y=0$ the shear stress is greater than at any other point of the section, that is, at points in the neutral line. The shear stress is 0 at C. Again, the end sections of the beam have greatest shear. A student has much food for thought in this result (5). It is interesting to find the directions and amounts of the principal stresses at every point of the beam, that is, the interfaces at right angles to one another at any point, across which there is only compression or only tension without tangential stress.

We have been considering a rectangular section. The student ought to work exercises on other sections as soon as he is able to integrate by with regard to y in (1) where b is any function of y. He will notice that $\int_{AE}^{AC} by.dy$ is equal to the area of $EHCHE$, fig. 47, multiplied by the distance of its centre of gravity from AA.

Taking a flanged section the student will find that f is small in the flanges and gets greater in the web. Even in a rectangular section f became rapidly smaller further out from the neutral line, but now to obtain it we must divide by the breadth of the section and this breadth is comparatively so great in the flanges that there is practically no shearing there, the shear being confined to the web; whereas in the web itself f does not vary very much. The student already knows that it is our usual custom to calculate the areas of the flanges or top and bottom booms of a girder as if they merely resisted compressive and tensile forces, and the web or the diagonal bracing as if it merely resisted shearing. He will note that the shear in a section is great only where $\frac{dM}{dx}$ or rather $\frac{d}{dx}\left(\frac{M}{I}\right)$ is great. But inasmuch as in Art. 66 we saw that $\frac{dM}{dx}=S$, the total shearing force at the section, there is nothing very extraordinary in finding that the actual shear stress anywhere in the section depends upon $\frac{dM}{dx}$. In a uniformly loaded beam $\frac{dM}{dx}$ is greatest at the ends and gets less and less towards the middle and then changes sign, hence the bracing of a girder loaded mainly with its own weight is much slighter in the middle than at the ends.

Deflection of Beams. If a bending moment M acts at a section of a beam, the part of length δx gets the strain-energy $\frac{1}{2}\frac{M^2 . \delta x}{EI}$, because $M . \delta x/EI$ is the angular change (see Art. 26), and therefore the whole strain-energy in a beam due to bending moment is

$$\frac{1}{2E}\int \frac{M^2}{I} . dx \quad\text{..............................} (6).$$

If f is a shear stress, the shear strain-energy per unit volume is $f^2/2N$...(7), and by adding we can therefore find its total amount for the whole beam.

By equating the strain-energy to the loads multiplied by half the displacements produced by them we obtain interesting relations. Thus in the case of a beam of length l, of rectangular section, fixed at one end and loaded at the other with a load W; at the distance x from the end, $M = Wx$ and the energy due to bending is

$$\frac{1}{2E}\int_0^l \frac{W^2x^2}{I} . dx = W^2l^3/6EI \quad\text{....................} (8).$$

The above expression (5) gives for the shearing stress

$$f = \frac{1}{b}\frac{6}{d^3}\left(\tfrac{1}{4}d^2 - y^2\right) W \quad\text{......................} (9).$$

The shear strain-energy in the elementary volume $b . \delta x . \delta y$ is $b . \delta x . \delta y . f^2/2N$. Integrating this with regard to y from $-\frac{1}{2}d$ to $+\frac{1}{2}d$ we find the energy in the slice between two sections to be

$$3W^2l . \delta x/5Nbd,$$

so that the shear strain-energy in the beam is $3W^2l/5Nbd$...(10).

If now the load W produces the deflection z at the end of the beam the work done is $\frac{1}{2}Wz$...(11).

Equating (11) to the sum of (8) and (10) we find

$$z = \frac{Wl^3}{3EI} + \tfrac{6}{5} . \frac{Wl}{Nbd} \quad\text{......................} (12).$$

Note that the first part of this due to bending is the deflection as calculated in Art. 60, Example I. We believe that the other part due to shearing has never before been calculated.

If the deflection due to bending is z_1 and to shearing is z_2,

$$z_1/z_2 = 10Nl^2/3Ed^2.$$

Taking $N = \frac{2}{5}E$ as being fairly correct, then $z_1/z_2 = 4l^2/3d^2$. If a beam is 10 inches deep, when its length is 8·6 inches the deflections due to bending and shear are equal; when its length is 86 inches, the deflection due to bending is 100 times that due to shear; when its length is 0·79 inch, the deflection due to bending is only 1/100th of that due to shear. Probably however our assumed laws of bending do not apply to so short a beam.

69. Springs which Bend. Let fig. 49 show the centre line of a spring *fixed* at A, loaded at B with a small load W in the direction shown. To find the amount of yielding at B. The load and the deflection are supposed to be very small. Consider the piece of spring bounded by cross sections at P and Q. Let $PQ = \delta s$, the length of the spring between B and P being called s.

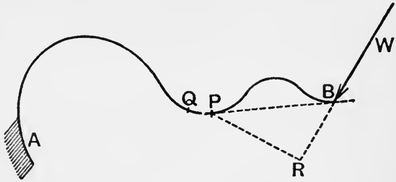

Fig. 49.

The bending moment at P is $W.PR$ or $W.x$ if x is the length of the perpendicular from P upon the direction of W. Let BR be called y. Consider first that part of the motion of B which is due to the change of shape of QP *alone*; that is, imagine AQ to be perfectly rigid and PB a rigid pointer. The section at Q being fixed, the section at P gets an angular change equal to $\delta s \times$ the change of curvature there, or $\delta s \dfrac{M}{EI}$ or $\dfrac{\delta s \,.\, Wx}{EI}$ ...(1), where E is Young's modulus and I is the moment of inertia of the cross section. The motion of B due to this is just the same as if PB were a straight pointer; in fact the pointer PB gets this angular motion and the motion of B is this angle, multiplied by the straight distance PB or

$$\frac{\delta s \,.\, Wx}{EI} \,.\, PB \quad \text{......................(2).}$$

Now how much of B's motion is in the direction of W?

It is its whole motion $\times \dfrac{PR}{PB}$ or $\times \dfrac{x}{PB}$ and hence B's motion in the direction of W is

$$\frac{\delta s \,.\, Wx^2}{EI} \quad \text{........................(3).}$$

Similarly B's motion at right angles to the direction of W is

$$\frac{\delta s \,.\, W \,.\, xy}{EI} \quad \text{..........................(4).}$$

In the most general cases, it is easy to work out the integrals of (3) and (4) graphically.

We usually divide the whole length of the spring from B to A into a large number of equal parts so as to have all the values of δs the same, and then we may say (s being the whole length of the spring) that we have to multiply $\frac{s \,.\, W}{E}$ upon the average values of $\frac{x^2}{I}$ and $\frac{xy}{I}$ for each part. In a well made spring if b is the breadth of a strip at right angles to the paper and t its thickness so that $I = \frac{1}{12}bt^3$ we usually have the spring equally ready to break everywhere or $\frac{6\,Wx}{bt^2} = f$, a constant. When this is the case (3) and (4) become

$$\frac{2f \,.\, \delta s}{E} \,.\, \frac{x}{t} \text{ and } \frac{2f \,.\, \delta s}{E} \,.\, \frac{y}{t}.$$

And if the strip is constant in thickness, varying in breadth in proportion to x, then

$$\frac{2f \,.\, \delta s}{Et} \,.\, x \text{ is (3) and } \frac{2f \,.\, \delta s}{Et} \,.\, y \text{ is (4).}$$

If $\bar{x}$ and $\bar{y}$ are the x and y of the centre of gravity of the curve (see Art. 48)

$\frac{2fs\bar{x}}{Et}$ is the total yielding parallel to W,

$\frac{2fs \,.\, \bar{y}}{Et}$ is the total yielding at right angles to W.

70. Exercises. The curvature of a curve is

$$\frac{1}{r} = \frac{d^2y}{dx^2} \Big/ \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}. \qquad \text{(See Art. 224.)}$$

When the equation to a curve is given it is easy to find

$\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and calculate $\frac{1}{r}$ where r is the radius of curvature. This is mere exercise work and it is not necessary to prove beforehand that the formula for the curvature is correct.

1. Find the curvature of the parabola $y = ax^2$ at the point $x = 0$, $y = 0$.

2. The equation to the shape of a beam, loaded uniformly and supported at the ends is $y = \frac{w}{48EI}(3l^2x^2 - 2x^4)$, see Art. 60, where the origin is at the middle of the beam; l is the whole length of the beam, w is the load per unit length, E is Young's modulus for the material and I is the moment of inertia of the cross section. Take $l = 200$, $w = 5$, $E = 29 \times 10^6$, $I = 80$, find the curvature where $x = 0$. Show that in this case $\left(\frac{dy}{dx}\right)^2$ may be neglected, in comparison with 1, and that really the curvature is represented by $\frac{d^2y}{dx^2}$. Show that the bending moment of the above beam is $M = \frac{w}{8EI}(l^2 - 4x^2)$. Show that this is greatest at the middle of the beam.

3. Find the curvature of the curve $y = a \log x + bx + c$ at the point where $x = x_1$.

71. Force due to Pressure of Fluids. *Exercise* 1. Prove that if p, the pressure of a fluid, is constant, the resultant of all the pressure forces on the plane area A is Ap and acts through the centre of the area.

2. The pressure in a liquid at the depth h being wh, where w is the weight of unit volume, what is the total force due to pressure on any immersed plane area? Let DE be the surface from which the depth h is measured and where the

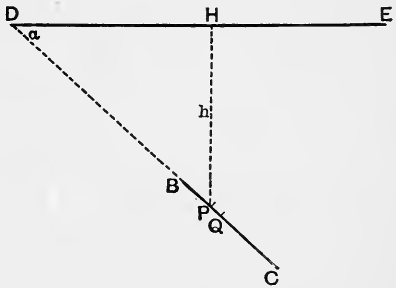

Fig. 50.

pressure is 0. Let BC be an edge view of the area; imagine its plane produced to cut the level surface of the liquid DE in D. Let the angle EDC be called α. Let the distance DP be called x and let DQ be called $x + \delta x$, and let the breadth of the area at right angles to the paper at P be called z. On the strip of area $z \,.\, \delta x$ there is the pressure wh if h is PH the depth of P, and $h = x \sin \alpha$, so that the pressure force on the strip is

$$wx \,.\, \sin \alpha \,.\, z \,.\, \delta x,$$

and the whole force is $F = w \sin \alpha \int_{DB}^{DC} x \,.\, z \,.\, dx$ ....(1).

Also if this resultant acts at a point in the area at a distance X from D, taking moments about D,

$$FX = w \sin \alpha \int_{DB}^{DC} x^2 \,.\, z \,.\, dx \text{..............(2).}$$

Observe in (1) that $\int_{DB}^{DC} x \,.\, z \,.\, dx = A\bar{x}$,

if A is the whole area and $\bar{x}$ is the distance of its centre of gravity from D. Hence, the **average pressure over the area is the pressure at the centre of gravity of the area.**

Observe in (2) that $\int_{DB}^{DC} x^2 z \,.\, dx = I$ the moment of inertia of the area about D. Letting $I = k^2 A$, where k is called the radius of gyration of the area about D, we see that

$$F = w \sin \alpha \,.\, A\bar{x}, \quad FX = w \sin \alpha \,.\, Ak^2.$$

Hence $X = \frac{k^2}{\bar{x}}$ ...(3), **the distance from D at which the resultant force acts.**

Example. If $DB = 0$ and the area is rectangular, of constant breadth b; then

$$I = b \int_0^{DC} x^2 \,.\, dx = \frac{b}{3} DC^3,$$

and $A = b \,.\, DC$ so that $k^2 = \frac{1}{3} DC^2$. Also $\bar{x} = \frac{1}{2} DC$. Hence $X = \frac{2}{3} DC$, that is, the resultant force acts at $\frac{2}{3}$ of the way

down the rectangle from D to C and the average pressure is the pressure at a point half way down.

It is an easily remembered relation that we find in (3). For if we have a compound pendulum, whose radius of gyration is k and if $\bar{x}$ is the distance from the point of support to its centre of gravity and if X is the distance to its point of percussion, we have the very same equation (3). Again, if X is the length of the simple pendulum which oscillates in exactly the same time as the compound one, we have again this same relation (3). These are merely mathematical helps to the memory, for the three physical phenomena have no other relation to one another than a mathematical one.

Whirling Fluid.

72. Suppose a mass of fluid to rotate like a rigid body about an axis with the angular velocity of α radians per second. Let OO be the axis. Let P be a particle weighing w lbs. Let $OP = x$.

The centrifugal force in pounds of any mass is the mass multiplied by the square of its angular velocity, multiplied by x. Here the mass is $\frac{w}{g}$ and the centrifugal force is $\frac{w}{g}\alpha^2 x$.

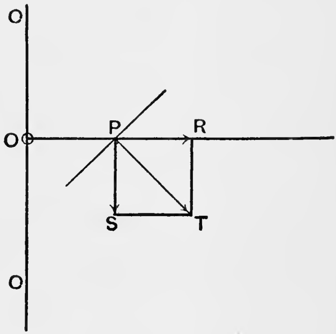

Fig. 51.

Make PR represent this to scale and let PS represent w the weight, to the same scale, then the resultant force, represented by PT, is easily found and the angle RPT which PT makes with the horizontal. Thus $\tan RPT = w \div \frac{w}{g}\alpha^2 x$ or $g \div \alpha^2 x$, being independent of w; we can therefore apply our results to heterogeneous fluid. Now if y is the distance of the point P above some datum level, and we imagine a curve drawn through P to which PT is (at P) tangential, and if at

every point of the curve its direction (or the direction of its tangent) represents the direction of the resultant force; if such a curve were drawn its slope $\frac{dy}{dx}$ is evidently $-\frac{g}{\alpha^2 x}$ and its equation is $\qquad y = -\frac{g}{\alpha^2}\log x + \text{constant}$ ....(1).

The constant depends upon the datum level from which y is measured. This curve is called a line of force. Its direction at any place shows the direction of the total force there. We see that it is a logarithmic curve.

Level Surfaces. If there is a curve to which PT is a

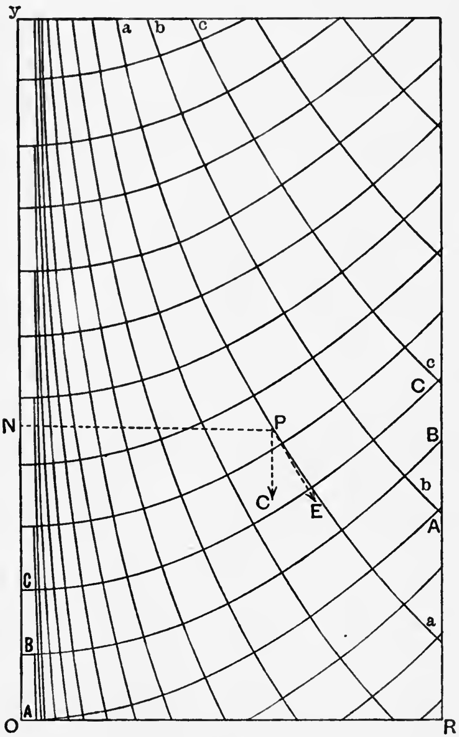

Fig. 52.

normal at the point P, it is evident that its slope is positive and in fact

$$\frac{dy}{dx} = \frac{\alpha^2}{g} x,$$

so that the curve is $$y = \frac{\alpha^2}{2g} x^2 + \text{constant} \ldots\ldots\ldots\ldots (2),$$

the constant depending upon the datum level from which y is measured. This is a parabola, and if it revolves about the axis we have a paraboloid of revolution. Any surface which is everywhere at right angles to the force at every point is called a *level* surface and we see that the level surfaces in this case are paraboloids of revolution. These level surfaces are sometimes called equi-potential surfaces. It is easy to prove that the pressure is constant everywhere in such a surface and that it is a surface of equal density, so that if mercury, oil, water and air are in a whirling vessel, their surfaces of separation are paraboloids of revolution.

The student ought to draw one of the lines of force and cut out a template of it in thin zinc, OO being another edge. By sliding along OO he can draw many lines of Force. Now cut out a template for one of the parabolas and with it draw many level surfaces. The two sets of curves cut each other everywhere orthogonally. Fig. 52 shows the sort of result obtainable where aa, bb, cc are the logarithmic lines of force and AA, BB, CC are the level paraboloidal surfaces.

73. Motion of Fluid. If AB is a stream tube, in the

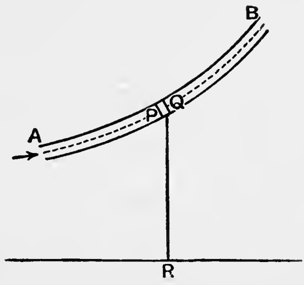

Fig. 53.

vertical plane of the paper, consider the mass of fluid between sections at P and Q of length δs feet along the stream, and cross-section a square feet, where a and δs are in the limit supposed to be infinitely small. Let the pressure at P be p lbs. per square foot, the velocity v feet per second, and let P be at the vertical height h feet above some datum level.

At Q let these quantities be $p + \delta p$, $v + \delta v$ and $h + \delta h$. Let the fluid weigh w lbs. per cubic foot.

Find the forces urging PQ along the stream, that is, forces parallel to the stream direction at PQ.

pa acts on one end P in the direction of motion, and $(p + \delta p)\,a$ acts at Q retarding the motion. The weight of the portion between P and Q is $a \,.\, \delta s \,.\, w$ and, as if on an inclined plane, its retarding component is

$$\text{weight} \times \frac{\text{height of plane}}{\text{length of plane}} \text{ or } a \,.\, \delta s \,.\, w \frac{\delta h}{\delta s}.$$

Hence we have altogether, accelerating the motion from P towards Q,

$$pa - (p + \delta p)\,a - a \,.\, \delta s \,.\, w \,.\, \frac{\delta h}{\delta s}.$$

But the mass is $\dfrac{a \,.\, \delta s \,.\, w}{g}$, and $\dfrac{dv}{dt}$ is its acceleration, and we have merely to put the force equal to $\dfrac{a \,.\, \delta s \,.\, w}{g} \,.\, \dfrac{dv}{dt}$. We have then, dividing by a,

$$-\delta p - \delta s \,.\, w \frac{\delta h}{\delta s} = \frac{\delta s \,.\, w}{g} \frac{dv}{dt}.$$

Now if δt be the time taken by a particle in going from P to Q, $v = \dfrac{\delta s}{\delta t}$ with greater and greater accuracy as δs is shorter and shorter. Also, the acceleration $\dfrac{dv}{dt}$ is more and more nearly $\dfrac{\delta v}{\delta t}$. (It is more important to think this matter out carefully than the student may at first suppose.)

Hence if δs is very small, $\delta s \,.\, \dfrac{dv}{dt} = \dfrac{\delta s}{\delta t} \,.\, \delta v = v \,.\, \delta v$, so that we have

$$\delta p + w \,.\, \delta h + \frac{w}{g} v \,.\, \delta v = 0 \text{ ....(1),}$$

or as we wish to accentuate the fact that this is more and more nearly true as δs is smaller and smaller, we may write it as

$$\frac{dp}{w}+dh+\frac{v}{g}\,.\,dv=0\;\ldots\ldots\ldots\ldots\ldots\ldots(2)^*,$$

or integrating, $$h+\frac{v^2}{2g}+\int\frac{dp}{w}=\text{constant}\;\ldots\ldots\ldots\ldots\ldots(2).$$

We leave the sign of integration on the $\frac{dp}{w}$ because w may vary. In a liquid where w is constant,

$$h+\frac{v^2}{2g}+\frac{p}{w}=\text{constant}\;\ldots\ldots\ldots\ldots\ldots(3).$$

74. In a gas, we have $w \propto p$ if the temperature could be kept constant, or we have the rule for adiabatic flow $w \propto p^{\frac{1}{\gamma}}$, where γ is the well-known ratio of the specific heats. In either of these cases it is easy to find $\int\frac{dp}{w}$ and write out the law. This law is of universal use in all cases where viscosity may be neglected and is a great guide to the Hydraulic Engineer.

Thus in the case of adiabatic flow $w=cp^{\frac{1}{\gamma}}$, the integral of $\frac{dp}{w}$ is

$\int\frac{dp}{cp^{1/\gamma}}$ or $\frac{1}{c}\int p^{-\frac{1}{\gamma}}\,.\,dp$ or $\frac{1}{c}\,\frac{\gamma}{\gamma-1}\,p^{1-\frac{1}{\gamma}}$, and hence, if s stand for $(\gamma-1)/\gamma$

we have $$h+\frac{v^2}{2g}+\frac{1}{cs}p^s=\text{constant}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

In a great many problems, changes of level are insignificant and we

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* After a little experience with quantities like δp &c., knowing as we do that the equations are not true unless δp, &c. are supposed to be smaller and smaller without limit and then we write their ratios as $\frac{dp}{dh}$, &c., we get into the way of writing dp, &c. instead of δp, &c.

Again, if $$f(x)\,.\,dx+F(y)\,dy+\phi(z)\,dz=0\;\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

then $$\int f(x)\,.\,dx+\int F(y)\,.\,dy+\int\phi(z)\,.\,dz=\text{a constant}\;\ldots\ldots\ldots(2).$$

There is no harm in getting accustomed to the integration of such an equation as (1), all across.

often use $v^2+\frac{2g}{cs}p^s=\text{constant}$......(4) for gases. Thus, if p_0 is the pressure and w_0 the weight of a cubic foot of gas inside a vessel at places where there is no velocity and if, outside an orifice, the pressure is p; the constant in (4) is evidently $0+\frac{2g}{cs}p_0^s$, and hence, outside the surface, $v^2=\frac{2g}{cs}(p_0^s-p^s)$......(5), and as c is $w_0 \div p_0^{\frac{1}{\gamma}}$ it is easy to make all sorts of calculations on the quantity of gas flowing per second.

Observe that if p is very little less than p_0, if we use the approximation $(1+a)^n=1+na$, when a is small, we find

$$v^2=\frac{2g}{w_0}(p_0-p)\text{..............................(6)},$$

a simple rule which it is well to remember in fan and windmill problems. In a Thomson Water Turbine the velocity of the rim of the wheel is the velocity due to half the total available pressure; so in an air turbine when there is no great difference of pressure, the velocity of the rim of the wheel is the velocity due to half the pressure difference.

Thus if p_0 of the supply is 7000 lbs. per square foot and if p of the exhaust is 6800 lbs. per square foot and if we take $w_0=0{\cdot}28$ lb. per cubic foot, the velocity of the rim V is, since the difference of pressure is 200 lbs. per square foot,

$$\sqrt{\frac{2g}{{\cdot}28}(100)}=151 \text{ feet per second.}$$

Returning to (5); neglecting friction, if there is an orifice of area A to which the flow is guided so that the streams of air are parallel, Q the volume flowing per second is $Q=vA$ and if the pressure is p, the weight of stuff flowing per second is

$$W=vAw,$$

or since $w=cp^{\frac{1}{\gamma}}$, and $w_0=cp_0^{\frac{1}{\gamma}}$,

$$W=vAw_0\left(\frac{p}{p_0}\right)^{\frac{1}{\gamma}}.$$

If the student will now substitute the value of v from (5) and put a for p/p_0 he will obtain

$$W=Aa^{\frac{1}{\gamma}}p_0\sqrt{\frac{2g\gamma}{\gamma-1}\,\frac{w_0}{p_0}\left(1-a^{\frac{\gamma-1}{\gamma}}\right)}\text{..................(7).}$$

Problem. Find p the outside pressure so that for a given inside pressure there may be a maximum flow.

It is obvious that as p is diminished more and more, v the velocity increases more and more and so does Q. But a large Q does not necessarily mean a large quantity of gas. We want W to be large. When is W a maximum? That is, what value of a in (7) will make

$$a^{\frac{2}{\gamma}}\left(1-a^{\frac{\gamma-1}{\gamma}}\right) \text{ or } a^{\frac{2}{\gamma}}-a^{1+\frac{1}{\gamma}}$$

a maximum? Differentiating with regard to a and equating to 0

$$\frac{2}{\gamma}a^{\frac{2}{\gamma}-1}-\left(1+\frac{1}{\gamma}\right)a^{\frac{1}{\gamma}}=0$$

dividing by $a^{\frac{1}{\gamma}}$ we find $a=\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{1-\gamma}}$.

Or
$$p=p_0\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}.$$

In the case of air $\gamma=1{\cdot}41$ and we find $p={\cdot}527p_0$.

That is, there is a maximum quantity leaving the vessel per second when the outside pressure is a little greater than half the inside pressure.

Problem. When p is indefinitely diminished what is v?

$$\text{Answer : } v=\sqrt{\frac{2g\gamma}{\gamma-1}\frac{p_0}{w_0}}.$$

This is greater than the velocity of sound in the ratio $\sqrt{\frac{2}{\gamma-1}}$, being 2·21 for air. That is, the limiting velocity in the case of air is 2413 feet per second $\times\sqrt{\frac{t}{273}}$, where t is the absolute temperature inside the vessel and there is a vacuum outside.

Students ought to work out as an example, the velocity of flow into the atmosphere.

Returning to equations (2) and (4), we assumed h to be of little importance in many gaseous problems of the mechanical engineer. But there are many physical problems in which it is necessary to take account of changes in level. For example if (2) is integrated on the assumption of constant temperature and we assume v to keep constant, we find that p diminishes as h increases according to the compound interest law considered in Chap. II. Again under the same condition as to v, but with the adiabatic law for w we find that p diminishes with h according to a law which may be stated as "the rate of diminution of temperature with h, is constant." These two propositions seem to belong more naturally to the subject matter of Chapter II.

75. A great number of interesting examples of the use of (2) might be given. It enables us to understand the flow of fluid from orifices, the action of jet pumps, the attraction of light bodies caused by vibrating tuning-forks, why some valves are actually sucked up more against their seats instead of being forced away by the issuing stream of fluid, and many other phenomena which are thought to be very curious.

Example 1. Particles of water in a basin, flowing very slowly towards a hole in the centre, move in nearly circular paths so that the velocity v is inversely proportional to the distance from the centre. Take $v=\frac{a}{x}$ where a is some constant and x is the radius or distance from the axis. Then (3) (Art. 73) becomes

$$h+\frac{a^2}{2gx^2}+\frac{p}{w}=C.$$

Now at the surface of the water, p is constant, being the pressure of the atmosphere, so that, there

$$h=c-\frac{a^2}{2gx^2},$$

and this gives us the shape of the curved surface. Assume c and a, any values, and it is easy to calculate h for any value of x and so plot the curve. This curve rotated about the axis gives the shape of the surface which is a surface of revolution.

Example 2. Water flowing spirally in a horizontal plane follows the law $v=\frac{b}{x}$ if x is distance from a central point. Note that $p=C_1-\frac{1}{2}\frac{w}{g}\frac{b^2}{x^2}$.

The ingenious student ought to study how p and v vary at right angles to stream lines. He has only to consider the equilibrium of an elementary portion of fluid PQ, fig. 53, subjected to pressures, centrifugal force and its own weight in a direction normal to the stream.

He will find that if $\frac{dp}{dr}$ means the rate at which p

varies in a direction of the radius of curvature away from the centre of curvature and if α is the angle QPR, fig. 53, the stream being in the plane of the paper, which is vertical,

$$\frac{dp}{dr} = \frac{w}{g}\frac{v^2}{r} - w \sin \alpha \text{ ....(1).}$$

If the stream lines are all in horizontal planes

$$\frac{dp}{dr} = \frac{w}{g}\frac{v^2}{r} \text{ ....(2).}$$

Example 3. Stream lines all circular and in horizontal planes in a liquid, so that h is constant.

If $v = \frac{b}{r}$, where b is a constant,

$$\frac{dp}{dr} = \frac{w}{g} \cdot \frac{b^2}{r^3},$$

$$p = -\tfrac{1}{2}\frac{w}{g}\frac{b^2}{r^2} + \text{constant} \text{ ....(3).}$$

We see therefore that the fall of pressure as we go outward is exactly the same as in the last example. Show that this law, $v = b/r$, must be true if there is no 'rotation' (See Example 5).

Example 4. Liquid rotates about an axis as if it were a rigid body, so that $v = br$, then

$$\frac{dp}{dr} = \frac{w}{g} b^2 r,$$

$$p = \frac{1}{2}\frac{w}{g} b^2 r^2 + c.$$

This shows the law of increase of pressure in the wheel of a centrifugal pump when full, but when delivering no water.

Exercise. The pressure at the inside of the wheel of a centrifugal pump is 2116 lbs. per sq. foot, the inside radius is 0·5 foot, the outside radius 1 foot. The angular velocity of the wheel is $b = 30$ radians per second; draw a curve showing the law of p and r from inside to outside when very little water is being delivered. If the water leaves the wheel by a spiral path, the velocity everywhere outside being

inversely proportional to r, draw also the curve showing the law of p in the whirlpool chamber outside.

Example 5. The expression

$$\frac{v^2}{2g} + \frac{1}{w} p + h = E,$$

which remains constant all along a stream line, may be called the total store of energy of 1 lb. of water in the stream if the motion is steady.

Now $\frac{dE}{dr} = \frac{1}{g} v \frac{dv}{dr} + \frac{1}{w} \frac{dp}{dr} + \frac{dh}{dr}$ becomes from equation (1),

$$\frac{dE}{dr} = \frac{2v}{g} \times \frac{1}{2} \left(\frac{v}{r} + \frac{dv}{dr} \right).$$

This expression $\frac{1}{2} \left(\frac{v}{r} + \frac{dv}{dr} \right)$ is called the "average angular velocity" or "**the rotation**" *or the 'spin'* of the liquid. Hence

$$\frac{dE}{dr} = \frac{2v}{g} \times \text{rotation}.$$

When liquid flows by gravity from a small orifice in a large vessel where, at a distance inside the orifice, the liquid may be supposed at rest, it is obvious the E is the same in all stream lines, so that $\frac{dE}{dr}$ is 0, and there is no 'rotation' anywhere.

If when water is flowing from an orifice in a vessel we can say that across some section of the stream the velocity is everywhere normal to the section and that the pressure is everywhere atmospheric, we can calculate the rate of flow. It is as well to say at once that we know of no natural foundation for these assumptions. However wrong the assumptions may be, there is no harm in using them in mere exercises on Integration. There being atmospheric pressure at the still water level, if v is the velocity at a point at the depth h, if a is an element of area of the section, $Q = \Sigma a \sqrt{2gh}$ the summation being effected over the whole section, Q being the volume flowing. Thus if the section is a vertical plane and if at the depth h it is of horizontal breadth z, through

the area $z.\delta h$ water is flowing with the velocity $\sqrt{2gh}$, so that $\sqrt{2gh}\,.\,z\,.\,\delta h$ is the elementary volume flowing per second, and if h_1 and h_2 are the depths of the highest and lowest points of the orifice, the total flow is $Q=\sqrt{2g}\int_{h_1}^{h_2} zh^{\frac{1}{2}}\,.\,dh.$

Example 6. Rectangular section, horizontal breadth b,

$$Q=\sqrt{2g}b\int_{h_1}^{h_2} h^{\frac{1}{2}}.dh=\tfrac{2}{3}b\sqrt{2g}\Big[h^{\frac{3}{2}}\Big]_{h_1}^{h_2}=\tfrac{2}{3}b\sqrt{2g}\,(h_2^{\frac{3}{2}}-h_1^{\frac{3}{2}}).$$

Example 7. Triangular section, angle at depth h_1, base horizontal of length b at depth h_2. Then within the limits of integration it will be found that $z=\dfrac{b}{h_2-h_1}(-h_1+h)$.

Hence $Q=\dfrac{b\sqrt{2g}}{h_2-h_1}\displaystyle\int(-h_1h^{\frac{1}{2}}+h^{\frac{3}{2}})\,dh=\dfrac{b\sqrt{2g}}{h_2-h_1}\Big[(-\tfrac{2}{3}h_1h^{\frac{3}{2}}+\tfrac{2}{5}h^{\frac{5}{2}})\Big]_{h_1}^{h_2}$.

If the ratio h_2/h_1 be called r, it will be found that

$$Q=\frac{bh_1^{\frac{3}{2}}}{r-1}\frac{\sqrt{2g}}{15}\{6r^{\frac{5}{2}}-10r^{\frac{3}{2}}+25\}.$$

When the student has practised integration in Chap. III., he may in the same way find the hypothetical flow through circular, elliptic and other sections.

Returning to the rectangular section, there is no case practically possible in which h_1 is 0, but as this is a mere mathematical exercise let us assume $h_1=0$, and we have $Q=\frac{2}{3}b\sqrt{2g}\,h_2^{\frac{3}{2}}$. Now further assume that if there is a rectangular sharp-edged notch through which water flows, its edge or sill being of breadth b and at the depth h_2, the flow through it is in some occult way represented by the above answer, multiplied by a fraction called a coefficient of contraction, then $Q=cb\sqrt{2g}h_2^{\frac{3}{2}}$. Such is the so-called theory of the flow through a rectangular gauge notch. A true theory was based by Prof. James Thomson on his law of flow from similar orifices, one of the very few laws which the hydraulic engineer has to depend upon. We are sorry to think that nearly all the mathematics to be found in standard treatises on Hydraulics is of the above character, that is, it has only an occult connection with natural phenomena.

76. Magnetic Field about a straight round wire. There are two great laws in Electrical Science. They concern the two circuits, the magnetic circuit and the electric circuit, which are always linked through one another.

I. **The line integral** (called the Gaussage whatever the unit may be) **of Magnetic Force round any closed curve, is equal to the current** [multiplied by 4π if the current is in what is called absolute C.G.S. units (curious kind of absolute unit that needs a multiplier in the most important of all laws); multiplied by $4\pi/10$ if the current is in commercial units called Amperes].

II. **The line integral** (called the Voltage whatever the unit may be) **of Electromotive Force round any closed curve is equal to the magnetic current** (really, rate of change of induction) which is enclosed. [If the induction is in absolute C.G.S. units, we have absolute Voltage in C.G.S.; if the induction is in Webers the Voltage is in Volts.

We are to remember that in a **non-conducting medium** the voltage in any circuit produces electric displacement, and the rate of change of this is current, and we deal with this exactly as we deal with currents in conducting material. When we deal with the phenomena in very small portions of space we speak of electric and magnetic currents per unit area, in which case the line integrals are called **'curls.'** Leaving out the annoying 4π or $4\pi/10$, we say, with Mr Heaviside, "The electric current is the curl of the magnetic force and the magnetic current is the negative curl of the electric force." When we write out these two statements in mathematical language, we have the two great Differential Equations of Electrical Analysis.

The Electrical Engineer is continually using these two laws. Many examples will be given, later, of the use of the second law. We find it convenient to give here the following easy example of the first law.

Field about a round wire. A straight round wire of radius a centimetres conveys current C $\left[\text{or } A \text{ amperes, so that } C = \frac{A}{10}\right]$.

If H is the magnetic force at a distance r from the centre of the wire, the Gaussage round the circle of radius r is $H \times 2\pi r$, because H is evidently, from symmetry, the same all round. Hence, as Gaussage $= 4\pi C$,

$$H = 4\pi C \div 2\pi r = \frac{2C}{r} \left[\text{or } \frac{2}{r}\frac{A}{10}\right].$$

Inside the wire, a circle of radius r encloses the total current $\frac{r^2}{a^2}C$, and hence H inside the wire at a distance r from the axis is

$$\frac{2rC}{a^2} \left[\text{or } \frac{2r}{10}\frac{A}{a^2}\right].$$

If BC is a cross section of the round wire of radius a, and if OD is any plane through the axis O of the wire, and

$$OP = r,\ OQ = r + \delta r:$$

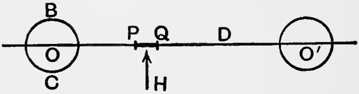

Fig. 54.

then through the strip of area PQ, which is l centimetres long at right angles to the paper, and δr wide, area $l.\delta r$, there is the induction H per sq. cm. [We take the permeability as 1. If μ is the magnetic permeability of the medium, the induction is $\beta = \mu H$ per sq. cm.], or $H.l.\delta r$ through the strip of area in question. If there are two parallel wires with opposite currents, and if OD is the plane through the axes of the two wires, the fields due to the two currents add themselves together. If O' is the centre of the other wire, the total H at P is $2C\left(\frac{1}{OP} + \frac{1}{O'P}\right)$.

77. Self-Induction of two parallel wires. Let the radius of each wire be a, and the distance between their centres b, the length of each being l between two planes at right angles to both. The wires are supposed to be parts of two infinite wires, to get rid of difficulties in imagining the circuit completed at the ends.

The total induction from axis to axis is the sum of the two amounts, $4l\int_a^b \frac{C.dr}{r}$ from the outside of each wire to the

axis of the other and $4l\int_0^a \frac{rC}{a^2}\,dr$ from the axis of each wire to its own surface. This is

$$2lC\left\{2\log\frac{b}{a}+1\right\}, \text{ or } \frac{2lA}{10}\left\{2\log\frac{b}{a}+1\right\} \text{ in absolute units.}$$

Dividing by 10^8 we have it in commercial units.

This total field when the current is 1, is the self-induction L of the circuit (we imagine current to be uniformly distributed over the section of the wire), and

$$\frac{L}{l}=2\left\{\log\frac{b^2}{a^2}+1\right\} \text{ in C.G.S. units,}$$

$$\frac{L}{l}=\frac{2}{10^9}\left\{\log\frac{b^2}{a^2}+1\right\}^*$$

in Henries per centimetre length of the two circuits.

78. Function of Two Independent Variables. Hitherto we have been studying a function of one variable, which we have generally called x. In trying to understand Natural Phenomena we endeavour to make one thing only vary. Thus in observing the laws of gases, we measure the change of pressure, letting the volume only change, that is, keeping the temperature constant, and we find $p \propto \frac{1}{v}$. Then we keep v constant and let the temperature alter, and we find $p \propto t$ (where $t=\theta^\circ\,\mathrm{C.}+274$). After

* Notice that one Henry is 10^9 absolute units of self-induction; our commercial unit of Induction called the Weber is 10^8 absolute units of Induction.

The Henry suits the law: $\text{Volts}=RA+L\frac{dA}{dt}$,

The Weber suits $\text{Volts}=RA+N\,.\,\frac{dI}{dt}$,

where R is in ohms, A amperes, L Henries, N the number of turns in a circuit, I Weber's of Induction.

In Elementary Work such as is dealt with in this book, I submit to the use of 4π and the difficulties introduced by the unscientific system now in use. In all my higher work with students, such as may be dealt with in a succeeding volume, I always use now the rational units of Heaviside and I feel sure that they must come into general use.

much trial we find, for one pound of a particular gas, the law $pv = Rt$ to be very nearly true, R being a known constant.

Now observe that any one of the three, p, v or t, is a function of the other two; and in fact **any values whatsoever** may be given to two, and the other can then be found.

Thus $$p = R\frac{t}{v} \qquad \text{(1)},$$

we can say that p is a function of the two *independent* variables t and v.

If any particular values whatsoever of t and v be taken in (1) we may calculate p. Now take new values, say $t + \delta t$ and $v + \delta v$, where δt and δv are perfectly independent of one another, then

$$p + \delta p = R\frac{t + \delta t}{v + \delta v} \text{ and } \delta p = R\frac{t + \delta t}{v + \delta v} - R\frac{t}{v}.$$

We see therefore that the change δp can be calculated if the independent changes δt and δv are known.

When all the changes are considered to be smaller and smaller without limit, we have an easy way of expressing δp in terms of δt and δv. It is

$$\delta p = \left(\frac{dp}{dt}\right)\delta t + \left(\frac{dp}{dv}\right)\delta v \qquad \text{(2)}.$$

This will be proved presently, but the student ought first to get acquainted with it. Let him put it in words and compare his own words with these: "The whole change in p is made up of two parts, 1st the change which would occur in p if v did not alter, and 2nd the change in p if t did not alter." The first of these is $\delta t \times$ the rate of increase of p with t when v is constant, or as we write it $\left(\frac{dp}{dt}\right)\delta t$, and the second of these is $\delta v \times$ the rate of increase of p with v if t is constant.

This idea is constantly in use by every practical man. It is only the algebraic way of stating it that is unfamiliar, and a student who is anxious to understand the subject will manufacture many familiar examples of it for himself.

Thus when one pound of stuff which is defined by its p, v and t, changes in state, the change is completely defined by any two of the changes δp and δv, or δv and δt, or δp and δt, because we are supposed to know the *characteristic* of the stuff, that is, the law connecting p, v and t.

Now the heat δH given to the stuff in any small change of state can be calculated from any two of δv, δt and δp, and all the answers ought to agree. As we wish to accentuate the fact that the changes are supposed to be exceedingly small we say

$$\left.\begin{aligned} dH &= k\,.\,dt + l\,.\,dv \\ &= K\,.\,dt + L\,.\,dp \\ &= P\,.\,dp + V\,.\,dv \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where the coefficients k, l, K, L, P and V are all functions of the state of the stuff, that is of any two of v, t and p. Notice that $k\,.\,dt$ is the heat required for a small change of state, defined by its change of temperature, if the volume is kept constant: hence k is called the specific heat at constant volume. In the same way K is called the specific heat at constant pressure. As for l and L perhaps they may be regarded as some kinds of latent heat, as the temperature is supposed to be constant.

These coefficients are not usually constant, they depend upon the state of the body. The mathematical proof that if δH can be calculated from δt and δv, then $dH = k\,.\,dt + l\,.\,dv$, where k and l are some numbers which depend upon the state of the stuff, is this:—If δH can be calculated, then $\delta H = k\,.\,\delta t + l\,.\,\delta v + a(\delta t)^2 + b(\delta v)^2 + c(\delta t\,.\,\delta v) + e(\delta t)^3 +$ terms of the third and higher degrees in δt and δv, where k, l, a, b, c, e &c. are coefficients depending upon the state of the body. Dividing by either δt or δv all across, and assuming δt and δv to diminish without limit, the proposition is proved.

Illustration. Take it that for one pound of *Air*, (1) is true and R is, say, 96, p being in lb. per sq. foot and v in cubic feet.

As $p = 96\,\frac{t}{v}$, $\quad \left(\frac{dp}{dt}\right) = \frac{96}{v}$, $\quad \left(\frac{dp}{dv}\right) = -\frac{96t}{v^2} = -\frac{p}{v}$.

Hence, from (2), $\quad \delta p = \frac{96}{v}\,.\,\delta t - \frac{p}{v}\,.\,\delta v$ ....(4).

Example. Let $t = 300$, $p = 2000$, $v = 14{\cdot}4$.

If t becomes 301 and v becomes 14·5 it is easy to show that p will become 1992·83. But we want to find the change in pressure, using (2) or rather (4),

$$\delta p = \frac{96}{14{\cdot}4} \times 1 - \frac{2000}{14{\cdot}4} \times {\cdot}1 = -\,7{\cdot}22 \text{ lb. per sq. ft.},$$

whereas the answer ought to be $-\,7{\cdot}17$.

Now try $\delta t = \cdot 1$ and $\delta v = \cdot 01$ and test the rule. Again, try $\delta t = \cdot 01$ and $\delta v = \cdot 001$, or take any other very small changes. In this way the student will get to know for himself what the rule (1) really means. It is only true when the changes are supposed to be smaller and smaller without limit.

Here is an exceedingly interesting exercise:—Suppose we put $\delta p = 0$ in (2). We see then a connection between δt and δv *when these changes occur at constant pressure.* Divide one of them by the other; we have $\frac{\delta v}{\delta t}$ when p is constant, or rather

$$\left(\frac{dv}{dt}\right) = -\frac{\left(\frac{dp}{dt}\right)}{\left(\frac{dp}{dv}\right)} \text{ ....(5).}$$

At first sight this minus sign will astonish the student and give him food for thought, and he will do well to manufacture for himself illustrations of (5). Thus to illustrate it with $pv = Rt$. Here

$$\left(\frac{dv}{dt}\right) = \frac{R}{p}, \quad \left(\frac{dp}{dt}\right) = \frac{R}{v}, \quad \left(\frac{dp}{dv}\right) = -\frac{Rt}{v^2} \text{ or } -\frac{p}{v},$$

and (5) states the truth that

$$\frac{R}{p} = -\frac{R}{v} \div \left(-\frac{p}{v}\right).$$

The student cannot have better exercises than those which he will obtain by expressing δv in terms of δt and δp, or δt in terms of δp and δv for any substance, and illustrating his deductions by the stuff for which $pv = Rt$. †

79. Further Illustrations. In (3) we have the same answer whether we calculate from dt and dv, or from dt and dp, or from dp and dv. Thus for example,

$$k \,.\, dt + l \,.\, dv = K \,.\, dt + L \,.\, dp \text{ .... (6).}$$

We saw that $dp = \left(\frac{dp}{dt}\right) dt + \left(\frac{dp}{dv}\right) dv$, and hence substituting this for dp in (6) we have

$$k \,.\, dt + l \,.\, dv = K \,.\, dt + L\left(\frac{dp}{dt}\right) dt + L\left(\frac{dp}{dv}\right) dv.$$

This is true for any independent changes dt and dv; let $dv=0$, and again let $dt=0$, and we have

$$k=K+L\left(\frac{dp}{dt}\right) \qquad (7),$$

$$l=L\left(\frac{dp}{dv}\right) \qquad (8).$$

Again, in (6) substitute $dv=\left(\frac{dv}{dt}\right)dt+\left(\frac{dv}{dp}\right)dp$, and we have

$$k\,.\,dt+l\left(\frac{dv}{dt}\right)dt+l\left(\frac{dv}{dp}\right)dp=K\,.\,dt+L\,.\,dp.$$

Equating coefficients of dt and of dp as before we have

$$k+l\left(\frac{dv}{dt}\right)=K \qquad (9),$$

$$l\left(\frac{dv}{dp}\right)=L \qquad (10).$$

Again, putting $k\,.\,dt+l\,.\,dv=P\,.\,dp+V\,.\,dv$, and substituting

$$dp=\left(\frac{dp}{dt}\right)dt+\left(\frac{dp}{dv}\right)dv,$$

we have $$k\,.\,dt+l\,.\,dv=P\left(\frac{dp}{dt}\right)dt+P\left(\frac{dp}{dv}\right)dv+V\,.\,dv,$$

and $$k=P\left(\frac{dp}{dt}\right) \qquad (11),$$

also $$l=P\left(\frac{dp}{dv}\right)+V \qquad (12).$$

Again, putting $K\,.\,dt+L\,.\,dp=P\,.\,dp+V\,.\,dv$, and substituting

$$dt=\left(\frac{dt}{dp}\right)dp+\left(\frac{dt}{dv}\right)dv,$$

we have $$K\left(\frac{dt}{dp}\right)dp+K\left(\frac{dt}{dv}\right)dv+L\,.\,dp=P\,.\,dp+V\,.\,dv,$$

and $$K\left(\frac{dt}{dp}\right)+L=P \qquad (13),$$

$$K\left(\frac{dt}{dv}\right)=V \qquad (14).$$

The relations (7), (8), (9), (10), (11), (12), (13) and (14) which are not really all independent of one another (and indeed we may get others in the same way) **are obtained merely mathematically** and without assuming any laws of Thermodynamics. We have called

H, heat; t temperature &c., but we need not, unless we please, attach any physical meaning to the letters.

The relations are true for any substance. Find what they become in the case of the stuff for which **pv = Rt** (the mathematical abstraction called a perfect gas). We know that

$$\left(\frac{dp}{dt}\right)= \frac{R}{v}, \text{ so that (7) becomes } k=K+L\frac{R}{v} \quad \text{......(7)*},$$

$$\left(\frac{dp}{dv}\right)=-\frac{p}{v}, \text{ so that (8) becomes } l=-L\frac{p}{v} \quad \text{............ (8)*},$$

$$\left(\frac{dv}{dt}\right)= \frac{R}{p}, \text{ so that (9) becomes } k+l\frac{R}{p}=K \quad \text{......... (9)*},$$

$$\left(\frac{dv}{dp}\right)=-\frac{v}{p}, \text{ so that (10) becomes } -l\frac{v}{p}=L \quad \text{......... (10)*},$$

$$\left(\frac{dp}{dt}\right)= \frac{R}{v}, \text{ so that (11) becomes } k=P\frac{R}{v} \quad \text{............(11)*},$$

$$\left(\frac{dp}{dv}\right)=-\frac{p}{v}, \text{ so that (12) becomes } l=-P\frac{p}{v}+V \quad \text{.....(12)*}.$$

It is evident that these are not all independent; thus using (10)* in (9)* we obtain (7)*.

80. Another Illustration. The Elasticity of our stuff is defined, see Art. 58, as

$$e=-v\frac{dp}{dv}.$$

Now if t is constant, we shall write this $e_t=-v\left(\frac{dp}{dv}\right)$, or the elasticity when the temperature remains constant.

If it is the *adiabatic* elasticity e_H which we require, we want to know the value of $\frac{dp}{dv}$ when the stuff neither loses nor gains heat. In the last expression of (3) put $dH=0$, and the ratio of our dp and our dv will then be just what is wanted or $\left(\frac{dp}{dv}\right)_H=-\frac{V}{P}$, the H being affixed to indicate that H is constant or that the stuff neither loses nor gains heat. Hence $e_H=v\frac{V}{P}$.

Thus
$$\frac{e_H}{e_t}=-\frac{V}{P}\div\left(\frac{dp}{dv}\right).$$

Taking V from (14) Art. 79 and P from (11),

$$\frac{e_H}{e_t}=-\frac{K\left(\frac{dt}{dv}\right)\left(\frac{dp}{dt}\right)}{k\left(\frac{dp}{dv}\right)},$$

but we have already seen as in (5) that $\left(\frac{dp}{dv}\right) \div \left(\frac{dp}{dt}\right) = -\left(\frac{dt}{dv}\right)$ and

hence **for any substance** $\frac{\mathbf{e_H}}{\mathbf{e_t}} = \frac{\mathbf{K}}{\mathbf{k}}$ .... (15).

This ratio of the two specific Heats is usually denoted by the letter γ. Note that neither of the two laws of Thermodynamics nor a Scale of temperature is referred to in this proof.

81. **General Proof.** If u is a function of x and y, we may write the statement in the form $u = f(x, y)$. Take particular values of x and y and calculate u. Now take the values $x + \delta x$ and $y + \delta y$, where δx and δy are perfectly independent of one another, and calculate the new u, call it $u + \delta u$. Now subtract and we can only *indicate* our result by

$$\delta u = f(x + \delta x,\ y + \delta y) - f(x,\ y).$$

Adding and subtracting the same thing $f(x,\ y + \delta y)$ we have

$$\delta u = f(x + \delta x,\ y + \delta y) - f(x,\ y + \delta y) + f(x,\ y + \delta y) - f(x,\ y).$$

This is the same as

$$\delta u = \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x}\,\delta x + \frac{f(x, y + \delta y) - f(x, y)}{\delta y}\,\delta y \ldots (16).$$

Now if δx and δy be supposed to get smaller and smaller without limit, the coefficient of δy

$$\text{or } \frac{f(x,\ y + \delta y) - f(x,\ y)}{\delta y} \text{ becomes } \frac{df(x,\ y)}{dy} \text{ or } \left(\frac{du}{dy}\right),$$

the x being constant. In fact this is our definition of a differential coefficient (see Art. 20, Note). Again, the coefficient of δx becomes the limiting value of $\frac{f(x + \delta x,\ y) - f(x,\ y)}{\delta x}$, because δy is evanescent. Writing then u instead of $f(x,\ y)$ we have

$$\mathbf{du} = \left(\frac{\mathbf{du}}{\mathbf{dx}}\right)\mathbf{dx} + \left(\frac{\mathbf{du}}{\mathbf{dy}}\right)\mathbf{dy} \ldots\ldots\ldots\ldots\ldots\ldots (17).$$

Thus if $u = ax^2 + by^2 + cxy$, $du = (2ax + cy)\,dx + (2by + cx)\,dy$.

82. Notice that although we may have

$$dz = M \,.\, dx + N \,.\, dy \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18),$$

where M and N are functions of x and y; it does not follow that z is a function of x and y. For example, we had in (3)

$$dH = k \,.\, dt + l \,.\, dv,$$

where k and l are functions of t and v. Now **H the total heat** which has been given to a pound of stuff is *not* a function of v and t; it **is not a function of the state of the stuff.** Stuff may

receive enormous quantities of heat energy, being brought back to its original state again, and yet not giving out the same amounts of heat as it received. The first law of Thermodynamics states however that if $\mathbf{dE = dH - p \cdot dv}$, where $p \cdot dv$ is the mechanical work done, we can give to E the name **Intrinsic Energy** because it is something which is a function of the state of the stuff. It always comes back to the same value when the stuff returns to the same state.

Our E is then some function of t and v, or of t and p, or of p and v, but H is not!

The second law of Thermodynamics is this:—If dH be divided by t where t is θ°C. + 274, θ°C. being measured on the perfect gas thermometer, and if $\frac{dH}{t}$ be called $d\phi$, then ϕ is called the **Entropy** of the stuff, and ϕ is a function of the state of the stuff.

83. It is very important, if

$$dz = M \cdot dx + N \cdot dy \quad \text{.......................} \quad (18),$$

where M and N are functions of x and y, to know when z is a function of x and y. If this is the case, then (18) is really

$$dz = \left(\frac{dz}{dx}\right) dx + \left(\frac{dz}{dy}\right) dy,$$

that is, M is $\left(\frac{dz}{dx}\right)$ and N is $\left(\frac{dz}{dy}\right)$,

and hence

$$\left(\frac{\mathbf{dM}}{\mathbf{dy}}\right) = \left(\frac{\mathbf{dN}}{\mathbf{dx}}\right) \quad \text{.........................} \quad (19),$$

because it is known that $\frac{d^2z}{dy \cdot dx} = \frac{d^2z}{dx \cdot dy}$ *.

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* Proof that $\frac{\mathbf{d^2u}}{\mathbf{dy \cdot dx}} = \frac{\mathbf{d^2u}}{\mathbf{dx \cdot dy}}$.

We gave some illustrations of this in Art. 31, and if the student is not yet familiar with what is to be proved, he had better work more examples, or work the old ones over again.

Let $u = f(x, y)$;

$\left(\frac{du}{dx}\right)$ is the limiting value of $\frac{f(x+\delta x, y) - f(x, y)}{\delta x}$ as δx gets smaller and smaller. Now this is a function of y, so $\frac{d}{dy}\left(\frac{du}{dx}\right)$ or $\frac{d^2u}{dy \cdot dx}$ is, by our definition of a differential coefficient, the limiting value of

$$\frac{1}{\delta y}\left\{\frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x} - \frac{f(x+\delta x, y) - f(x, y)}{\delta x}\right\}$$

as δy and δx get smaller and smaller.

Here we have an exceedingly important rule:—If

$$dz = M \,.\, dx + N \,.\, dy \quad \text{..........................(18)},$$

and if z is a function of x and y (another way of saying that z is a function of x and y is to say that $dz = M \,.\, dx + N \,.\, dy$ is a *complete differential*), then

$$\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right)^{*} \quad \text{..........................(19)}.$$

Working the reverse way, we find that $\frac{d^2u}{dx \,.\, dy}$ is the limiting value of

$$\frac{1}{\delta x}\left\{\frac{f(x+\delta x,\ y+\delta y) - f(x+\delta x,\ y)}{\delta y} - \frac{f(x,\ y+\delta y) - f(x,\ y)}{\delta y}\right\}$$

as δy and δx get smaller and smaller. Now it is obvious that these two are the same for all values of δx and δy, and we assume that they remain the same in the limit.

* $$M \,.\, dx + N \,.\, dy \quad \text{......................................(1)},$$

where M and N are functions of x and y, **can always be multiplied by some function of x and y which will make it a complete differential.** This multiplier is usually called an integrating factor. For, whatever functions of x and y, M and N may be, we can write

$$\frac{dy}{dx} = -\frac{M}{N} \quad \text{......................................(2)},$$

and this means that there is *some* law connecting x and y. Call it

$$F(x,\ y) = c, \text{ then } \left(\frac{dF}{dx}\right) + \left(\frac{dF}{dy}\right)\frac{dy}{dx} = 0 \quad \text{...............(3)},$$

and as $\frac{dy}{dx}$ from (3) is the same as in (2) it follows that $\left(\frac{dF}{dx}\right) \div \left(\frac{dF}{dy}\right) = \frac{M}{N}$, and hence $\left(\frac{dF}{dx}\right) = \mu M$, $\left(\frac{dF}{dy}\right) = \mu N$, where μ is a function of x and y or else a constant.

Multiplying (1) by μ we evidently get

$$\left(\frac{dF}{dx}\right)dx + \left(\frac{dF}{dy}\right)dy \quad \text{..........................(4)},$$

and this is a complete differential. It is easy to show that not only is there an integrating factor μ but that there are an infinite number of them. As containing one illustration of the importance of this proposition I will state the steps in the **proof** which we have **of the 2nd law of Thermodynamics.**

1. We have shown that for any substance, of which the state is defined by its t and v,

$$dH = k \,.\, dt + l \,.\, dv \quad \text{..............................(5)},$$

where k and l are functions of t and v.

Observe that t may be measured on any curiously varying scale of temperature whatsoever. We have just proved that there is some function μ of t and

84. **The First Law of Thermodynamics** is this: If $dE = dH - p \,.\, dv$, or $dE = k \,.\, dt + (l-p)\, dv$, then dE is a **complete differential** that is, E returns to its old value when

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v by which if we multiply (5) all across we obtain a complete differential; indeed there are an infinite number of such functions. Then calling the result $d\phi$,

$$d\phi = \mu \,.\, dH = \mu k \,.\, dt + \mu l \,.\, dv \quad\text{.........................(6).}$$

Let us see if it is possible to find such a value of μ that it is a function of t only. If so, as the differential coefficient of μk with regard to v (t being supposed constant) is equal to the differential coefficient of μl with regard to t (v being supposed constant),

$$\mu \left(\frac{dk}{dv}\right)_t = l\,\frac{d\mu}{dt} + \mu\left(\frac{dl}{dt}\right)_v,$$

or

$$\left(\frac{dk}{dv}\right)_t = \left(\frac{dl}{dt}\right)_v + \frac{l}{\mu}\cdot\frac{d\mu}{dt} \quad\text{........ ....(7).}$$

But the first law of Thermodynamics (see Art. 84) gives us

$$\left(\frac{dk}{dv}\right)_t = \left(\frac{dl}{dt}\right)_v - \left(\frac{dp}{dt}\right) \quad\text{........................(8),}$$

and hence

$$\frac{1}{l}\left(\frac{dp}{dt}\right) = -\frac{1}{\mu}\cdot\frac{d\mu}{dt} \quad\text{............................ (9).}$$

This then is the condition that $\mu \,.\, dH$ is a complete differential, μ being a function of temperature only. Obviously for any given substance (9) will give us a value of μ which will answer; but what we really want to know is whether there is a value of μ which will be the same for all substances.

2. Here is the proof that there is such a value. I need not here give to students the usual and well-known proof that all reversible heat engines working between the temperatures t and $t - \delta t$ have the same efficiency. Now let $ABCD$ be a figure showing with infinite magnification an elementary Carnot cycle. Stuff at A at the temperature $t - \delta t$; AI shows the volume and AK the pressure. Let AD be the isothermal for $t - \delta t$ and BC the isothermal for t, AB and CD being adiabatics.

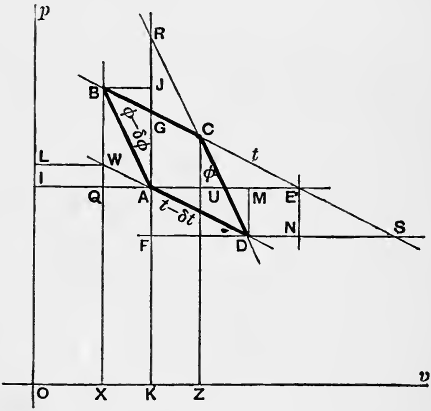

Fig. 55

Notice carefully that the distance AG or WB (W is in DA produced to meet the ordinate at B) is $(dp/dt)\,\delta t$.

Now the area of the parallelogram $ABCD$ which represents the work done, is $BW \times XZ$ (if parallelograms on the same base and between the same parallels be drawn, this will become clear). Call XZ by the symbol δv (the increase of volume in going along the isothermal from B to C), and we see that

the nett work done in the Carnot cycle is $(dp/dt)\,\delta t\,.\,\delta v$. Now the Heat t and v return to their old values, (or another way of putting it is that $\int dE$ for a complete cycle is 0).

We have seen that the differential coefficient of k with regard to v, t being constant, is equal to the differential coefficient of $l-p$ with regard to t, v being constant, or

$$\left(\frac{dk}{dv}\right)_t = \left(\frac{dl}{dt}\right)_v - \left(\frac{dp}{dt}\right) \text{ .... (20).}$$

This statement, which is true for any kind of stuff, is itself sometimes called the first law of Thermodynamics.

The Second Law of Thermodynamics is this; $\frac{dH}{t}$ or

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taken in at the higher temperature is, from (3), Art. 78, equal to $l\,.\,\delta v$ and hence $\frac{\text{nett work}}{\text{Heat}} = \text{efficiency} = \frac{1}{l}\left(\frac{dp}{dt}\right)\delta t$... (10), and this is the same for all substances.

As it is the same for all substances, let us try to find its value for any one substance. A famous experiment of Joule (two vessels, one with gas at high pressure, the other at low pressure with stopcock between, immersed in a bath all at same temperature; after equalization of pressure in the vessels, the temperature of the bath keeps its old value) showed that in gases, the intrinsic energy is very nearly constant at constant temperature, or what is the same thing, that l in gases is very nearly equal to p, and it is also well known that in gases at constant volume, p is a linear function of the temperature. Whether there really is an actual substance possible for which this is absolutely true, is a question which must now be left to the higher mathematicians, but we assume that there is such a substance and in it

$$\frac{1}{l}\left(\frac{dp}{dt}\right) = \frac{1}{p}\left(\frac{dp}{dt}\right) = \frac{1}{\theta+274} \text{ ....(11),}$$

if θ is the Centigrade reading on the Air Thermometer. If then we take $t=\theta+274$ as our scale of temperature and (11) as the universal value of $\frac{1}{l}\left(\frac{dp}{dt}\right)$, then, from (9), $\frac{1}{t} = -\frac{1}{\mu}\cdot\frac{d\mu}{dt}$, or $\frac{dt}{t} = -\frac{d\mu}{\mu}$ or $\log t + \log \mu = $ a constant, or $\mu = \frac{c}{t}$, where c is any constant. This being an integrating factor for (5), we usually take unity as the value of c or $\mu = \frac{1}{t}$ as Carnot's function.

It is not probable that, even if there is one which is independent of p or v, there really is so simple a multiplier as $\frac{1}{\theta+274}$ (where θ is the Centigrade temperature on the air thermometer) or that there is such a substance as we have postulated above. Calling our divisor t the **absolute temperature**, we believe that for ordinary values of θ, t is $\theta+274$, and the greater θ is, the more correctly is t represented by $\theta+274$; but when θ is very small, in all probability the absolute temperature is a much more complicated function of θ. The great discoverers of the laws of Thermodynamics never spoke of $-274°$ C. as the absolute zero of temperature.

$d\phi = \frac{k}{t} \cdot dt + \frac{l}{t} \cdot dv \ldots (21)$, is a complete differential, and hence the differential coefficient of $\frac{k}{t}$ with regard to v, t being considered constant, is equal to the differential coefficient of $\frac{l}{t}$ with regard to t, v being constant †.

Hence $$\frac{1}{t}\left(\frac{dk}{dv}\right)_t = \frac{t\left(\frac{dl}{dt}\right)_v - l}{t^2},$$

or $$\left(\frac{dk}{dv}\right)_t = \left(\frac{dl}{dt}\right)_v - \frac{l}{t} \quad \ldots\ldots\ldots\ldots (22).$$

This statement, which is true for any kind of stuff, is itself sometimes called the second law of Thermodynamics.

Combining (20) and (22), we have for any stuff

$$\left(\frac{\mathbf{dp}}{\mathbf{dt}}\right) = \frac{\mathbf{l}}{\mathbf{t}} \quad \ldots\ldots\ldots\ldots (23),$$

a most important law. †

Applying these to the case of a perfect gas we find that (23) becomes $\frac{l}{t} = \frac{R}{v}$, or $l = \frac{Rt}{v}$, or $l = p$.....................(24).

Hence (20) is $\left(\frac{dk}{dv}\right)_t = 0$. It is not of much importance perhaps, practically, but a student ought to study this last statement as an exercise. k is, for any substance, a function of v and t, and here we are told that for a perfect gas, however k may behave as to temperature, it does not change with change of volume. Combining (24) with (9)* &c. (p. 141), already found, we have $K - k = R$, and as Regnault found that K is constant for air and other gases, k is also constant, so that

$$l = p,\ L = -v,\ P = \frac{v}{\gamma - 1},\ V = \frac{p\gamma}{\gamma - 1}, \text{ where } \gamma = \frac{K}{k}.$$

We can now make *exact* calculations on the Thermodynamics of a perfect gas if we know K and R.

85. The statements of (3) Art. 78 become for a pound of **perfect gas**

$$\left.\begin{aligned} \mathbf{dH} &= \mathbf{k\ .\ dt + p\ .\ dv} \\ &= \mathbf{K\ .\ dt - v\ .\ dp} \\ &= \frac{\mathbf{v}}{\gamma - \mathbf{1}}\mathbf{dp} + \frac{\mathbf{p}\gamma}{\gamma - \mathbf{1}}\mathbf{dv}. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (1).$$

† The rule for finding the differential coefficient of a quotient is given in Art. 197.

I often write this last in the shape $\frac{1}{\gamma-1} d(pv)+p \,.\, dv$ ....(2),

also $\qquad dE=k\,.\,dt$, or $E=kt+\text{constant}$..................(3).

It is easy to obtain from this other forms of E in terms of p and v. To the end of this article, I consider the stuff to be a perfect gas.

Example 1. $\qquad d\phi=k\,.\,\frac{dt}{t}+\frac{p}{t}\,.\,dv$, or as $\frac{p}{t}=\frac{R}{v}$

$$d\phi=k\frac{dt}{t}+\frac{R}{v}\,.\,dv.$$

Hence, integrating,

$$\phi=k\log t+R\log v+\text{constant, or } \phi=\log t^k v^R+\text{constant}......(4).$$

Again $\qquad d\phi=\frac{K}{t}\,.\,dt-\frac{v}{t}\,.\,dp$, but $\frac{v}{t}=\frac{R}{p}$.

Hence $\qquad d\phi=\frac{K}{t}\,dt-\frac{R}{p}\,dp.$

Integrating

$$\phi=K\log t-R\log p+\text{constant, or } \phi=\log t^K p^{-R}+\text{constant}...(5).$$

Substituting for t its value $\frac{pv}{R}$ we have (5) becoming

$$\phi=\log p^k v^K+\text{constant} \;........................(6).$$

The adiabatic law, or ϕ constant, may be written down at once. Reducing from the above forms we find

$$tv^{\gamma-1}=\text{constant},$$

or $$t^{\frac{\gamma}{1-\gamma}}p=\text{constant},$$

or $$pv^{\gamma}=\text{constant}.$$

Students may manufacture other interesting exercises of this kind for themselves.

Example 2. A pound of gas in the state p_0, v_0, t_0 receives the amount of heat H_{01}, what change of state occurs? We get our information from (1).

I. Let **the volume** v_0 **keep constant.** Then $dH=k\,.\,dt$ from (1).

The integral of this between t_0 and t_1 is $H_{01}=k(t_1-t_0)$, and we may calculate the rise of temperature to t_1.

Or again, $dH=\frac{v_0}{\gamma-1}\,dp.$

Hence, the integral, or $H_{01}=\frac{v_0}{\gamma-1}(p_1-p_0)$, and we may calculate the rise of pressure.

II. Let p_0 the **pressure, keep constant.**

$$dH=K\,.\,dt, \text{ hence } H_{01}=K(t_1-t_0).$$

Again $$dH=\frac{p_0\gamma}{\gamma-1}dv, \text{ hence } H_{01}=\frac{p_0\gamma}{\gamma-1}(v_1-v_0).$$

III. **At constant temperature.**

$dH=p\,.\,dv$ or $H_{01}=\int_{v_0}^{v_1}p\,.\,dv=W$, the work done by the gas in expanding.

IV. **Under any conditions** of changing pressure and volume.

$$H_{01}=k(t_1-t_0)+\text{work done.}$$

Also from (2), $H_{01}=\frac{1}{\gamma-1}(p_1v_1-p_0v_0)+\text{work done.}$

If $H=0$, the work done $=k(t_0-t_1)$

We often write the last equation of (1) in the convenient shape

$$\frac{\mathbf{dH}}{\mathbf{dv}}=\frac{\mathbf{1}}{\gamma-\mathbf{1}}\left\{\mathbf{v}\frac{\mathbf{dp}}{\mathbf{dv}}+\gamma\mathbf{p}\right\} \quad \text{.....................(7).}$$

If in this we have no reception of heat,

or $$\frac{dH}{dv}=0, \text{ then } v\frac{dp}{dv}+\gamma\,.\,p=0,$$

or $$\frac{dp}{p}+\gamma\frac{dv}{v}=0 \text{ or, integrating, } \log p+\gamma\log v=\text{constant,}$$

or $pv^\gamma=\text{constant}$.

This is the adiabatic law again.

Example 3. In a well known **gas or oil engine cycle of**

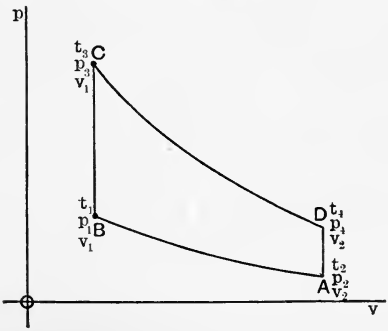

Fig. 56.

operations, a pound of gas at p_2, v_2, t_2, indicated by the point A is compressed adiabatically to B, where we have p_1, v_1, t_1. The work done *upon* the gas is evidently (from IV.) $k(t_1-t_2)$, being indeed the *gain of intrinsic energy.*

Heat given at constant volume from B to C where we have p_3, v_1, t_3 is $H=k(t_3-t_1)$.

Work done in adiabatic expansion $CD=k(t_3-t_4)$.

Nett work done = work in CD − work in AB=

$$W=k(t_3+t_2-t_1-t_4),$$

$$\frac{W}{H}=\text{efficiency } e=\frac{t_3+t_2-t_1-t_4}{t_3-t_1}=1-\frac{t_4-t_2}{t_3-t_1} \text{ ....(8).}$$

But we saw that along an adiabatic $tv^{\gamma-1}$ is constant, and hence

$$t_2v_2^{\gamma-1}=t_1v_1^{\gamma-1},$$
$$t_4v_2^{\gamma-1}=t_3v_1^{\gamma-1}.$$

From this it follows that $\frac{t_4}{t_3}=\frac{t_2}{t_1}=\left(\frac{v_1}{v_2}\right)^{\gamma-1}$, and each of these $=\frac{t_4-t_2}{t_3-t_1}$. Using this value in (8) we have

$$\textbf{efficiency}=\mathbf{1}-\left(\frac{\mathbf{v_1}}{\mathbf{v_2}}\right)^{\gamma-1} \text{ ....(9),}$$

a formula which is useful in showing the gain of efficiency produced by diminishing the clearance v_1.

Students will find other good exercises in other cycles of gas engines.

Change of State.

86. Instead of using equations (3) Art. 78, let us get out equations specially suited to change of state. Let us consider one pound of substance, m being vapour, $1-m$ being liquid (or, if the change is from solid to liquid, m liquid, $1-m$ solid), and let

s_2 = cubic feet of one pound of vapour,
s_1 = „ „ of one pound of liquid,
p = pressure, t temperature. p is a function of t only.

If v is the volume of stuff in the mixed condition,

$$v=ms_2+(1-m)s_1$$
$$=(s_2-s_1)m+s_1, \text{ or } v=mu+s_1 \text{ ....(1),}$$

if we write u for s_2-s_1.

When heat dH is given to the mixture, consider that t and m alter. In fact, take t and m as independent variables, noting that t and m define the state. If σ_2 and σ_1 be specific heats of vapour and liquid, when in the saturated condition (for example, σ_2 is the heat given to one pound of vapour to raise it one degree, its pressure rising at

the same time according to the proper law), then the m lb. of vapour needs the heat $m\sigma_2 . dt$, and the $1-m$ of liquid needs the heat $(1-m)\,\sigma_1 . dt$ and *also* if dm of liquid becomes vapour, the heat $L . dm$ is needed, if L is latent heat. Hence

$$dH = \{(\sigma_2 - \sigma_1)\, m + \sigma_1\}\, dt + L \,.\, dm \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

If E is the Intrinsic Energy, the first law of Thermodynamics gives

$$dE = dH - p \,.\, dv \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (3).$$

Now if m and t define the state, v must be a function of m and t, or

$$dv = \left(\frac{dv}{dt}\right) . \, dt + \left(\frac{dv}{dm}\right) dm.$$

Using this in (3) and (2) we find

$$dE = \left\{(\sigma_2 - \sigma_1)\, m + \sigma_1 - p\left(\frac{dv}{dt}\right)\right\} dt + \left\{L - p\left(\frac{dv}{dm}\right)\right\} dm \ \ldots (4).$$

Stating that this is a complete differential, or

$$\frac{d}{dm}\left\{(\sigma_2 - \sigma_1)\, m + \sigma_1 - p\left(\frac{dv}{dt}\right)\right\} = \frac{d}{dt}\left\{L - p\left(\frac{dv}{dm}\right)\right\},$$

we have, noting from (1), that $\left(\dfrac{dv}{dm}\right) = u$,

$$\frac{dL}{dt} + \sigma_1 - \sigma_2 = \frac{dp}{dt} \cdot \left(\frac{dv}{dm}\right), \text{ or } u\,\frac{dp}{dt} \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Now divide (2) by t and state that $d\phi = \dfrac{dH}{t}$ is a complete differential,

$$\frac{d}{dm}\left\{\frac{(\sigma_2 - \sigma_1)\, m + \sigma_1}{t}\right\} = \frac{d}{dt}\left(\frac{L}{t}\right)^* \ldots\ldots\ldots\ldots\ldots\ (6),$$

or

$$\frac{dL}{dt} + \sigma_1 - \sigma_2 = \frac{L}{t} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (7).$$

Hence, with (5) we have $\mathbf{\dfrac{L}{t} = u\,\dfrac{dp}{dt}}$ .... (8),

$$= (s_2 - s_1)\,\frac{dp}{dt}.$$

87. To arrive at the fundamental Equation (8) more rapidly. In fig. 57 we have an elementary Carnot cycle for one pound

* $\dfrac{d}{dt}\left(\dfrac{L}{t}\right) = \dfrac{t \,.\, \dfrac{dL}{dt} - L}{t^2}$ as will be seen later on when we have the rule for differentiating a quotient. But indeed we may as well confess that to understand this article on change of state, students must be able to perform differentiation on a product or a quotient.

of stuff. The co-ordinates of the point B are $FB=s_1$ the volume, and BG the pressure p of 1 pound of liquid. At constant temperatnre t, and

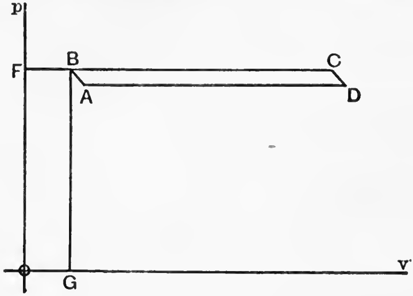

Fig. 57.

also constant pressure, the stuff expands until it is all vapour at $FC=s_2$; CD is adiabatic expansion to the temperature $t-\delta t$ at D. DA is isothermal compression at $t-\delta t$ and AB is the final adiabatic operation. The vertical height of the parallelogram is $\delta t\,\frac{dp}{dt}$, and its area, representing the nett work, is $\delta t\,.\,\frac{dp}{dt}(s_2-s_1)$. The heat taken in, in the operation BC is L, and the efficiency is $\delta t\,\frac{dp}{dt}(s_2-s_1)\div L$. But as it is a Carnot cycle this is equal to $\frac{\delta t}{t}$ and so we obtain (8). †

88. The Entropy. From (6) we find $\sigma_2-\sigma_1=t\,\frac{d}{dt}\left(\frac{L}{t}\right)$, and we can write (2) as

$$dH=\sigma_1 dt+t\,.\,d\left(\frac{mL}{t}\right) \quad\text{......................(9).}$$

Hence, the entropy $d\phi=\frac{dH}{t}=\frac{\sigma_1}{t}\,dt+d\left(\frac{mL}{t}\right)$,

or

$$\phi=\frac{mL}{t}+\int_{t_0}^{t}\frac{\sigma_1}{t}\,dt+\text{constant} \quad\text{..................(10).}$$

In the case of water, σ_1 is nearly constant, being Joule's equivalent. (We have already stated that all our heat is in work units), and

$$\phi=\frac{mL}{t}+\sigma_1\log\left(\frac{t}{t_0}\right)+\text{constant} \quad\text{..................(11).}$$

Hence the adiabatic law for water-steam is

$$\mathbf{\frac{mL}{t}+\sigma_1\log\frac{t}{t_0}=constant} \quad\text{..................(12).}$$

It is an excellent exercise for students to take a numerical example.

Let steam at 165°C. (or $t=439$) expand adiabatically to 85°C. (or $t=359$). Take $\sigma_1=1400$ and L in work units, or take $\sigma_1=1$ and take L in heat units. In any case, use a table of values of t and L.

1. At the higher $t_2=439$ let $m_2=\cdot 7$. (This is chosen at random.) Calculate m_1 at, say $t_1=394$, and also m_0 at $t_0=359$.

Perhaps we had better take L in heat units as the formula

$$L=796-\cdot 695t$$

is easily remembered.

Then (12) becomes

$$m_1\left(\frac{727}{t_1}-\cdot 695\right)+\log\frac{t_1}{t_0}=m_2\left(\frac{727}{t_2}-\cdot 695\right)+\log\frac{t_2}{t_0},$$

$$m_1=\frac{\log\frac{t_2}{t_1}+m_2\left(\frac{727}{t_2}-\cdot 695\right)}{\frac{727}{t_1}-\cdot 695}.$$

If we want m_0 we use t_0 instead of t_1.

Having done this, find the corresponding values of v. Now try if there is any law like

$$pv^s=\text{constant},$$

which may be approximately true as the adiabatic of this stuff. Repeat this, starting with $m_2=\cdot 8$ say, instead of $\cdot 7$.

The t, ϕ diagram method is better for bringing these matters most clearly before students, but one or two examples like the above ought to be worked.

89. When a complete differential du is zero, **to solve the equation** $du=0$. We see that in the case,

$$(x^2-4xy-2y^2)\,dx+(y^2-4xy-2x^2)\,dy=0,$$

we have a complete differential, because

$$\frac{d}{dy}(x^2-4xy-2y^2)=-4x-4y,$$

$$\frac{d}{dx}(y^2-4xy-2x^2)=-4y-4x,$$

so that they are equal. Hence it is of the form

$$\left(\frac{du}{dx}\right).\,dx+\left(\frac{du}{dy}\right)dy.$$

Integrating $x^2-4xy-2y^2$, since it is $\left(\frac{du}{dx}\right)$, with regard to

x assuming y constant, and adding, instead of a constant, an arbitrary function of y, we get u as

$$u = \tfrac{1}{3}x^3 - 2x^2y - 2y^2x + \phi(y).$$

To find $\phi(y)$, we know that $\left(\dfrac{du}{dy}\right) = y^2 - 4xy - 2x^2$.

Hence $$-2x^2 - 4yx + \frac{d}{dy}\phi(y) = y^2 - 4xy - 2x^2.$$

Hence $$\frac{d}{dy}\phi(y) = y^2 \text{ or } \phi(y) = \tfrac{1}{3}y^3.$$

Hence $$u = \tfrac{1}{3}x^3 - 2x^2y - 2xy^2 + \tfrac{1}{3}y^3 = c.$$

We have therefore solved the given differential equation when we put this expression equal to an arbitrary constant.

Solve in the same manner,

$$\left(1 + \frac{y^2}{x^2}\right)dx - 2\frac{y}{x}dy = 0. \quad \text{Answer } x^2 - y^2 = cx.$$

Solve $\dfrac{2x\,.\,dx}{y^3} + \left(\dfrac{1}{y^2} - \dfrac{3x^2}{y^4}\right)dy = 0.$ Answer $x^2 - y^2 = cy^3$.

Solve $\left(3x^2 + 3y - \dfrac{4}{x^3}\right)dx + \left(3x - \dfrac{8}{y^3} + 3y^2\right)dy = 0.$

Answer $x^5y^2 + x^2y^5 + 4x^2 + 2y^2 + 3x^3y^3 = cx^2y^2$.

90. In the general proof of (17) given in Art. 81, we assumed that **x and y were perfectly independent.** We may now if we please **make them depend either upon one another** or any third variable z. Thus if when any independent quantity z becomes $z+\delta z$, x becomes $x+\delta x$ and y becomes $y+\delta y$, of course u becomes $u+\delta u$. Let (16) Art. 81 be divided all across by δz, and let δz be diminished without limit, then (17) becomes

$$\frac{du}{dz} = \left(\frac{du}{dx}\right)\frac{dx}{dz} + \left(\frac{du}{dy}\right)\frac{dy}{dz} \quad \text{.......................(1).}$$

Thus let $$u = ax^2 + by^2 + cxy,$$

and let $$x = ez^n,\ y = gz^m.$$

Then $\left(\dfrac{du}{dx}\right) = 2ax + cy$, $\left(\dfrac{du}{dy}\right) = 2by + cx$, $\dfrac{dx}{dz} = nez^{n-1}$, $\dfrac{dy}{dz} = mgz^{m-1}$,

and consequently

$$\frac{du}{dz} = (2ax + cy)\,nez^{n-1} + (2by + cx)\,mgz^{m-1}.$$

In this we may, if we please, substitute for x and y in terms of z, and so get our answer all in terms of z.

This sort of example is rather interesting because it can be worked out in our earlier way. In the expression for u, substitute for x and y in terms of z, and we find $u=ae^2z^{2n}+bg^2z^{2m}+cegz^{n+m}$ and

$$\frac{du}{dz}=2nae^2z^{2n-1}+2mbg^2z^{2m-1}+(n+m)\,cegz^{n+m-1}.$$

It will be found that this is exactly the same as what was obtained by the newer method. The student can easily manufacture examples of this kind for himself.

For instance, let $y=uv$ where u and v are functions of x, then (1) tells us that

$$\frac{\mathbf{dy}}{\mathbf{dx}}=\mathbf{u}\frac{\mathbf{dv}}{\mathbf{dx}}+\mathbf{v}\frac{\mathbf{du}}{\mathbf{dx}},$$

a formula which is usually worked out in a very different fashion. See Art. 196.

In (1) if y is really a constant, the formula becomes

$$\frac{\mathbf{du}}{\mathbf{dz}}=\frac{\mathbf{du}}{\mathbf{dx}}\cdot\frac{\mathbf{dx}}{\mathbf{dz}},$$

which again is a formula which is usually worked out in a very different fashion. See Art. 198.

In (1) assume that $z=x$ and that y is a function of x, then

$$\frac{du}{dx}=\left(\frac{du}{dx}\right)+\left(\frac{du}{dy}\right)\frac{dy}{dx}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The student need not now be told that $\frac{du}{dx}$ is a very different thing from $\left(\frac{du}{dx}\right)$.

Example. Let $\qquad u=ax^2+by^2+cxy,$

and let $\qquad y=gx^m.$

Then $\left(\frac{du}{dx}\right)=2ax+cy,\ \left(\frac{du}{dy}\right)=2by+cx,\ \frac{dy}{dx}=mgx^{m-1}.$

Hence (2) is, $\quad \frac{du}{dx}=(2ax+cy)+(2by+cx)\,mgx^{m-1}.$

More directly, substituting for y in u, we have

$$u=ax^2+bg^2x^{2m}+cgx^{m+1},$$

$$\frac{du}{dx}=2ax+2mbg^2x^{2m-1}+(m+1)\,cgx^m,$$

and this will be found to be the same as the other answer.

If u is a function of three independent variables it is easy to prove, as in Art. 81, that

$$du=\left(\frac{du}{dx}\right)dx+\left(\frac{du}{dy}\right)dy+\left(\frac{du}{dz}\right)dz \quad\text{.................(3)}.$$

91. *Example.* **When a mass m is vibrating** with one degree of freedom under the control of a spring of stiffness a, so that if x is the displacement of the mass from its position of equilibrium, then ax is the force with which the spring acts upon the mass; we know that the potential energy is $\frac{1}{2}ax^2$ (see Art. 26), and if v is the velocity of the mass at the same time t, the kinetic energy is $\frac{1}{2}mv^2$, and we neglect the mass of the spring, then the total store of energy is

$$E=\tfrac{1}{2}mv^2+\tfrac{1}{2}ax^2.$$

When x is 0, v is at its greatest; when v is 0, x is at its greatest.

1. Suppose this store E to be constant and differentiate with regard to t, then

$$0=mv\frac{dv}{dt}+ax\frac{dx}{dt} \quad\text{..........................(1)},$$

or as v is $\frac{dx}{dt}$, writing $\frac{d^2x}{dt^2}$ for $\frac{dv}{dt}$ we have

$$\mathbf{\frac{d^2x}{dt^2}+\frac{a}{m}x=0} \quad\text{..........................(2)},$$

which is (see Art. 119) the well known law of simple harmonic motion.

2. If the total store of energy is not constant but diminishes at a rate which is proportional to the square of the velocity, as in the case of Fluid or Electromagnetic friction, that is, if $\frac{dE}{dt}=-Fv^2$ then (1) becomes $-Fv^2=mv\frac{dv}{dt}+ax\frac{dx}{dt}$, or (2) becomes

$$\mathbf{\frac{d^2x}{dt^2}+\frac{F}{m}\frac{dx}{dt}+\frac{a}{m}x=0} \quad\text{....................(3)}.$$

Compare (1) of Art. 142.

92. Similarly in a circuit with self induction L and resistance R, joining the coatings of a condenser of capacity K, if the current is C, and if the quantity of electricity in the condenser at time t is KV so that $C=-K\frac{dV}{dt}$, $\frac{1}{2}LC^2$ is called the kinetic energy of the system, and $\frac{1}{2}KV^2$ is the potential energy, and the loss of energy by the system per second is RC^2. So that if E is the store of energy at any instant

$$E=\tfrac{1}{2}LC^2+\tfrac{1}{2}KV^2,$$

$$\frac{dE}{dt}=-RC^2=LC\frac{dC}{dt}+KV\frac{dV}{dt},$$

or $$LC\frac{dC}{dt} - V\,.\,C + RC^2 = 0,$$

or $$L\frac{dC}{dt} - V + RC = 0,$$

or $$LK\frac{d^2V}{dt^2} + RK\frac{dV}{dt} + V = 0,$$

or $$\frac{d^2V}{dt^2} + \frac{R}{L}\frac{dV}{dt} + \frac{1}{LK}V = 0 \quad\text{.....................(4).}$$

Differentiating this all across and replacing $K\frac{dV}{dt}$ with C we have a similar equation in C. Compare (4) of § 145.

93. A mass m moving with velocity v has kinetic energy $\frac{1}{2}mv^2$. If this is its total store E,

$$E = \tfrac{1}{2}mv^2.$$

If E diminishes at a rate proportional to the square of its velocity as in fluid friction at slow speeds,

$$\frac{dE}{dt} = -Fv^2 = mv\frac{dv}{dt},$$

or $$\frac{dv}{dt} = -\frac{F}{m}v \quad\text{.....................................(5).}$$

We have a similar equation for the dying out of current in an electric conductor, $\frac{1}{2}LC^2$ being its kinetic energy, and RC^2 being the rate of loss of energy per second.

94. In (2) of Art. 90, assume that u is a constant and we find for example that if $u = f(x, y) = c$

$$\left(\frac{df(x, y)}{dx}\right) + \left(\frac{df(x, y)}{dy}\right)\frac{dy}{dx} = 0,$$

so that if $f(x, y) = c$ or $= 0$, we easily obtain $\frac{dy}{dx}$.

1. Thus if $x^2 + y^2 = c$, $2x + 2y\,.\,\frac{dy}{dx} = 0$, or $\frac{dy}{dx} = -\frac{x}{y}$.

2. Also if $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, $\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$, or $\frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y}$.

3. Again if $$u = Ax^m + By^n,$$

$$\frac{du}{dx} = mAx^{m-1} + nBy^{n-1}\frac{dy}{dx}.$$

Hence if $u = 0$, or a constant, we have

$$\frac{dy}{dx} = -\frac{mAx^{m-1}}{nBy^{n-1}}.$$

4. If
$$u=\frac{x^2}{a^2}+\frac{y^2}{b^2},$$
$$du=\frac{2x}{a^2}dx+\frac{2y}{b^2}dy.$$

5. If $x^3+y^3-3axy=b$, find $\frac{dy}{dx}$. Answer: $\frac{dy}{dx}=\frac{x^2-ay}{ax-y^2}$.

6. If
$$x\log y-y\log x=0,$$
$$\frac{dy}{dx}=\frac{y}{x}\left(\frac{x\log y-y}{y\log x-x}\right).$$

95. *Example.* Find the equations to the **tangent and normal to the ellipse** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, at the point x_1, y_1 on the curve.

$$\frac{x}{a^2}+\frac{y}{b^2}\frac{dy}{dx}=0, \text{ or at the point, } \frac{dy}{dx}=-\frac{b^2}{a^2}\frac{x_1}{y_1}.$$

Hence the equation to the tangent is

$$\frac{y-y_1}{x-x_1}=-\frac{b^2}{a^2}\frac{x_1}{y_1},$$

or
$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^{\,2}}{a^2}+\frac{y_1^{\,2}}{b^2},$$

and as x_1 and y_1 are the co-ordinates of a point in the curve, this is 1.

Hence the tangent is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1.$

The slope of the normal is $\frac{a^2}{b^2}\frac{y_1}{x_1}$, and hence the equation to the normal is $\frac{y-y_1}{x-x_1}=\frac{a^2}{b^2}\frac{y_1}{x_1}$.

APPENDIX TO CHAP. I.

Page 19. In an engineering investigation if one arrives at mathematical expressions which cannot really be thought about because they are too complicated, one can often get a simple empirical formula to replace them with small error within the limits between which they have to be used. Sometimes even such a simple expression as $a+bx$, or x^a will replace a complicated portion of an expression with small error. Expertness in such substitution is easily attained, especially in calculations where some of the terms can be expressed numerically or when one makes numerical experiments.

Exercise 1. The following observed numbers are known to follow a law like $y=a+bx$, but there are errors of observation. Find by the use of squared paper the most probable values of a and b.

x	2	3	$4\frac{1}{2}$	6	7	9	12	13
y	5·6	6·85	9·27	11·65	12·75	16·32	20·25	22·33

Ans. $y=2·5+1·5x$.

Exercise 2. The following numbers are thought to follow a law like $y=ax/(1+sx)$. Find by plotting the values of y/x and y on squared paper that these follow a law $y/x+sy=a$ and so find the most probable values of a and s.

x	·5	1	2	0·3	1·4	2·5
y	·78	·97	1·22	·55	1·1	1·24

Ans. $y=3x/(1+2x)$.

Exercise 3. If p is the pressure in pounds per square inch and if v is the volume in cubic feet of 1 lb. of saturated steam,

p	6·86	14·70	28·83	60·40	101·9	163·3	250·3
v	53·92	26·36	14·00	6·992	4·28	2·748	1·853

Plotting the common logarithms of p and v on squared paper test the truth of $pv^{1·0646}=479$.

Exercise 4. The following are results of experiments each lasting for four hours; I the indicated horse-power of an engine, transmitting B horse-power to Dynamo Machines which gave out E horse-power (Electrically), the weight of steam used per hour being W lb., the weight of coal used per hour being C lb. (the regulation of the engine was by changing the pressure of the steam). Show that, approximately, $W=800+21I$, $B=·95I-18$, $E=·93B-10$, $C=4·2I-62$.

I	B	E	W	C
190	163	143	4800	730
142	115	96	3770	544
108	86	69	3080	387
65	43	29	2155	218
19	0	0	1220	—

Page 34. It has been suggested to me by many persons that I ought to have given a proof without assuming the Binomial Theorem, and then the Binomial becomes only an example of Taylor's. In spite

of the eminence and experience of my critics, I believe that my method is the better—to tell a student that although I know he has not proved the Binomial, yet it is well to assume that he knows the theorem to be correct. The following seems to me the simplest proof which does not assume the Binomial.

Let $$y=x^n, \quad x+\delta x=x_1, \quad y+\delta y=y_1.$$

(1) Suppose n a positive integer; then

$$\frac{y_1-y}{x_1-x}=\frac{x_1{}^n-x^n}{x_1-x}=x_1{}^{n-1}+xx_1{}^{n-2}+\ldots+x^{n-1}.$$

In the limit, when δx is made smaller and smaller, until ultimately $x_1=x$, the left-hand side is $\frac{dy}{dx}$ and the right-hand side is $x^{n-1}+x^{n-1}+\ldots$ to n terms; so that $\frac{dy}{dx}=nx^{n-1}$.

(2) Suppose n a positive fraction, and put $n=\frac{l}{m}$ where l and m are positive integers. We have $\frac{y_1-y}{x_1-x}=\frac{x_1{}^{l/m}-x^{l/m}}{x_1-x}=\frac{z_1{}^l-z^l}{z_1{}^m-z^m}$ where $x^{\frac{1}{m}}=z$, $x_1=z_1{}^m$, and so on.

$$\frac{dy}{dx}=\text{limit of }\frac{(z_1-z)(z_1{}^{l-1}+z\,.\,z_1{}^{l-2}+\ldots\ldots+z^{l-1})}{(z_1-z)(z_1{}^{m-1}+z\,.\,z_1{}^{m-2}+\ldots\ldots+z^{m-1})}$$

$$=\frac{lz^{l-1}}{mz^{m-1}}=\frac{l}{m}z^{l-m}=nx^{\frac{l}{m}-1}=nx^{n-1}.$$

(3) Suppose n any negative number $=-m$ say, where m is positive, then noticing that $x_1{}^{-m}-x^{-m}=\frac{x^m-x_1{}^m}{x_1{}^m x^m}$,

we have $$\frac{x_1{}^{-m}-x^{-m}}{x_1-x}=-\frac{1}{x_1{}^{-m}x^m}\cdot\frac{x_1{}^m-x^m}{x_1-x}.$$

Now the limit of $\frac{x_1{}^m-x^m}{x_1-x}=mx^{m-1}$ by cases (1) and (2) whether m be integral or fractional.

$$\therefore\ \frac{dy}{dx}=-\frac{1}{x^{2m}}\cdot mx^{m-1}=-mx^{-m-1}=nx^{n-1}.$$

Thus we have shown that $\frac{d}{dx}(x^n)=nx^{n-1}$, where n is any constant, positive or negative, integral or fractional.

CHAPTER II.

$\mathbf{e^x}$ and sin x.

97. The Compound Interest Law. The solutions of an enormous number of engineering problems depend only upon our being able to differentiate x^n. I have given a few examples. Surely it is better to remember that the differential coefficient of x^n is nx^{n-1}, than to write hundreds of pages evading the necessity for this little bit of knowledge.

We come now to a very different kind of function, $\mathbf{e^x}$, where it is a constant quantity e (e is the base of the Napierian system of logarithms and is 2·7183) which is raised to a *variable* power. We calculate logarithms and exponential functions from series, and it is proved in Algebra that

$$e^x = 1 + x + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \frac{x^4}{1\,.\,2\,.\,3\,.\,4} + \&c.$$

The continuous product $1\,.\,2\,.\,3\,.\,4$ or 24 is denoted by $\underline{|4}$ or sometimes by $4\,!$

Now if we differentiate e^x term by term, we evidently obtain

$$0 + 1 + x + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \&c.,$$

so that the differential coefficient of e^x is itself e^x. Similarly we can prove that the differential coefficient of e^{ax} is ae^{ax}. This is the only function known to us whose rate of increase is proportional to itself; but there are a great many phenomena in nature which have this property. Lord Kelvin's way of putting it is that "they follow the compound interest law."

Notice that if $$\frac{dy}{dx} = ay \quad \text{.......................(1)},$$

that is, the rate of increase of y is proportional to y itself, then

$$y = be^{ax} \quad \text{.......................(2)},$$

where b is any constant whatsoever; b evidently represents the value of y when $x = 0$.

Here again, it will be well for a student to illustrate his proved rule by means of graphical and numerical illustrations. Draw the curve $y = e^x$, and show that its slope is equal to its ordinate. Or take values of x, say 2, 2·001, 2·002, 2·003, &c., and calculate the corresponding values of y using a table of Logarithms. (This is not a bad exercise in itself, for practical men are not always quick enough in their use of logarithms.) Now divide the increments of y by the corresponding increments of x. An ingenious student will find other and probably more complex ways of getting familiar with the idea. However complex his method may be it will be valuable to him, so long as it is his own discovery, but let him beware of irritating other men by trying to teach them through his complex discoveries.

98. It will perhaps lighten our study if we work out **a few examples of the Compound Interest Law.**

Our readers are either Electrical or Mechanical Engineers. If Electrical they must also be Mechanical. The Mechanical Engineers who know nothing about electricity may skip the electrical problems, but they are advised to study them; at the same time it is well to remember that one problem thoroughly studied is more instructive than thirty carelessly studied.

Example 1. **An electric condenser** of constant capacity K, fig. 58, discharging through great resistance R. If v is the potential difference (at a particular instant) between the condenser coatings, mark one coating as v and the other as 0 on your sketch, fig. 58. Draw an arrow-head representing the current C in the conductor; then $C = v \div R$.

But q the quantity of electricity in the condenser is Kv

and the rate of *diminution* of q per second or $-\frac{dq}{dt}$ or $-K\frac{dv}{dt}$ is the very same current. Hence

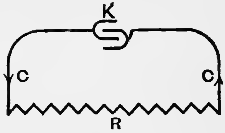

Fig. 58.

$$-K\frac{dv}{dt}=\frac{v}{R},$$

or
$$\frac{dv}{dt}=-\frac{1}{KR}v.$$

That is, the rate of *diminution* of v per second, is proportional to v, and whether it is a diminution or an increase we call this the compound interest law. We guess therefore that we are dealing with the exponential function, and after a little experience we see that any such example as this is a case of (1), and hence by (2)

$$v=be^{-\frac{1}{KR}t} \quad \text{.....................(3).}$$

It is because of this that we have the rule for finding the leakage resistance of a cable or condenser.

For
$$(\log b-\log v)=\frac{t}{KR}.$$

So that if v_1 is the potential at time t_1 and if v_2 is the potential at time t_2

$$KR(\log b-\log v_1)=t_1,$$

$$KR(\log b-\log v_2)=t_2.$$

Subtracting, $$KR(\log v_1-\log v_2)=t_2-t_1.$$

So that $$R=(t_2-t_1)/K\log\frac{v_1}{v_2}.$$

It is hardly necessary to say that the Napierian logarithm of a number n, $\log n$, is equal to the common logarithm $\log_{10} n$ multiplied by 2·3026.

Such an example as this, studied carefully step by step by an engineer, is worth as much as the careless study of twenty such problems.

Example 2. **Newton's law of cooling.** Imagine a body all at the temperature v (above the temperature of sur-

rounding bodies) to lose heat at a rate which is proportional to v.

Thus let $$\frac{dv}{dt} = -av,$$

where t is time. Then by (2)

$$v = be^{-at} \quad\text{.........................}(4),$$

or $$\log b - \log v = at.$$

Thus let the temperature be v_1 at the time t_1 and v_2 at the time t_2, then $\log v_1 - \log v_2 = a(t_2 - t_1)$, so that a can be measured experimentally as being equal to

$$\log \frac{v_1}{v_2} \div (t_2 - t_1).$$

Example 3. A rod (like a tapering winding rope or like **a pump rod** of iron, but it may be like a tie rod made of stone to carry the weight of a lamp in a church) tapers gradually because of its own weight, so that it may have everywhere in it exactly the same tensile stress f lbs. per square inch. If y is the cross section at the distance x from its lower end, and if $y + \delta y$ is its cross section at the distance $x + \delta x$ from its lower end, then $f . \delta y$ is evidently equal to the weight of the little portion between x and $x + \delta x$. This portion is of volume $\delta x \times y$, and if w is the weight per unit volume

$$f . \delta y = w . y . \delta x \text{ or rather } \frac{dy}{dx} = \frac{w}{f} y.$$

Hence as before, $$y = be^{\frac{w}{f}x} \quad\text{......................}(5).$$

If when $x = 0$, $y = y_0$, the cross section just sufficient to support a weight W hung on at the bottom (evidently $fy_0 = W$), then $y_0 = b$ because $e^0 = 1$.

It is however unnecessary to say more than that (5) is the law according to which the rod tapers.

Example 4. **Compound Interest.** £100 lent at 3 per cent. per annum becomes £103 at the end of a year. The interest during the second year being charged on the increased capital, the increase is greater the second year, and is greater and greater every year. Here the addition of interest due is made every twelve months; it might be

made every six or three months, or weekly or daily or every second. Nature's processes are, however, usually more continuous even than this.

Let us imagine compound interest to be added on to the principal continually, and not by jerks every year, at the rate of r per cent. per annum. Let P be the principal at the end of t years. Then δP for the time δt is $\frac{r}{100} P \,.\, \delta t$ or $\frac{dP}{dt} = \frac{r}{100} P$, and hence by (2) we have

$$P = be^{\frac{r}{100} t},$$

where $b = P_0$ the principal at the time $t = 0$.

Example 5. **Slipping of a Belt on a Pulley.** When students make experiments on this slipping phenomenon, they ought to cause the pulley to be fixed so that they may see the slipping when it occurs.

The pull on a belt at W is T_1, and this overcomes not only the pull T_0 but also the friction between the belt and the pulley. Consider the tension T in the belt at P, fig. 59, the angle QOP being θ; also the tension $T + \delta T$ at S, the angle QOS being $\theta + \delta\theta$.

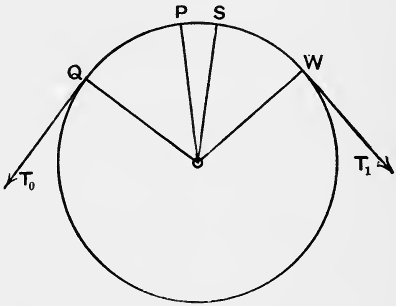

Fig. 59.

Fig. 60 shows part of OPS greatly magnified, $\delta\theta$ being very small. In calculating the force pressing the small portion of belt PS against the pulley rim, as we think of PS as a shorter and shorter length, we see that the resultant pressing force is $T \,.\, \delta\theta$*, so that $\mu \,.\, T \,.\, \delta\theta$ is

* When two equal forces T make a small angle $\delta\theta$ with one another, find their equilibrant or resultant. The three forces are parallel to the sides of an isosceles triangle like fig. 61, where $AB = CA$ represents T, where

Fig. 61.

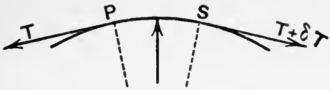

Fig. 60.

the friction, if μ is the coefficient of friction. It is this that δT is required to overcome. When $\mu . T . \delta\theta$ is exactly equalled by δT sliding is about to begin. Then $\mu . T . \delta\theta = \delta T$ or $\frac{dT}{d\theta} = \mu T$, the compound interest law. Hence $T = be^{\mu\theta}$. Insert now $T = T_0$ when $\theta = 0$, and $T = T_1$ when $\theta = QOW$ or θ_1, and we have $T_0 = b$, $T_1 = T_0 e^{\mu\theta_1}$.

In calculating the horse-power H given by a belt to a pulley, we must remember that $H = (T_1 - T_0) V \div 33000$, if T_1 and T_0 are in pounds and V is the velocity of the belt in feet per minute. Again, whether a belt will or will not tear depends upon T_1; from these considerations we have the well-known rule for belting.

Example 6. **Atmospheric Pressure.** At a place which is h feet above datum level, let the atmospheric pressure be p lbs. per sq. foot; at $h + \delta h$ let the pressure be $p + \delta p$ (δp is negative, as will be seen). The pressure at h is really greater than the pressure at $h + \delta h$ by the weight of air filling the volume δh cubic feet. If w is the weight in lbs. of a cubic foot of air, $-\delta p = w . \delta h$. But $w = cp$, where c is some constant **if the temperature is constant.** Hence $-\delta p = c . p . \delta h$...(1), or, rather $\frac{dp}{dh} = -cp$. Hence, as before, we have the compound interest law; the rate of fall of pressure as we go up or the rate of increase of pressure as we come down being proportional to the pressure itself. Hence $p = ae^{-ch}$, where a is some constant. If $p = p_0$, when $h = 0$, then $a = p_0$, so that the law is

$$p = p_0 e^{-ch} \quad \text{..........................(2).}$$

As for c we easily find it to be $\frac{w_0}{p_0}$, w_0 being the weight of a cubic foot of air at the pressure p_0. If t is the constant (absolute) temperature, and w_0 is now the weight of a cubic foot of air at 0° C. or 274° absolute, then c is $\frac{w_0}{p_0}\frac{274}{t}$.

$BAC = \delta\theta$ and BC represents the equilibrant. Now it is evident that as $\delta\theta$ is less and less, $BC \div AB$ is more and more nearly $\delta\theta$, so that the equilibrant is more and more nearly $T . \delta\theta$.

If w follows the adiabatic law, so that $pw^{-\gamma}$ is constant or $w = cp^{1/\gamma}$ where $\gamma = 1{\cdot}414$ for air. Then (1) becomes $-\delta p = cp^{1/\gamma}\,\delta h$ or $-\dfrac{\delta p}{p^{1/\gamma}} = c\delta h$ or rather $-\displaystyle\int \frac{dp}{p^{1/\gamma}} = ch$ or $\dfrac{-\gamma}{\gamma-1}\, p^{\frac{\gamma-1}{\gamma}} = ch + C$. If $p = p_0$ where $h = 0$, we can find C, and

we have $$p^{1-\frac{1}{\gamma}} = p_0^{1-\frac{1}{\gamma}} - \frac{\gamma-1}{\gamma}\, ch \quad\text{.................(3)},$$

as the more usually correct law for pressure diminishing upwards in the atmosphere.

Observe that when we have the adiabatic law $pv^\gamma = b$, a constant, and $pv = Rt$; it follows that the absolute temperature is proportional to $p^{1-\frac{1}{\gamma}}$.

So that (3) becomes
$$t = t_0 - \frac{1}{R}\frac{\gamma-1}{\gamma}\, h.$$

So that the rate of diminution of temperature is constant per foot upwards in such a mass of gas. Compare Art. 74, (4), if v is 0.

Example 7. **Fly-wheel stopped by a Fluid Frictional Resistance.**

Let α be its velocity in radians per second, I its moment of inertia. Let the resistance to motion be a torque proportional to the velocity, say $F\alpha$, then
$$F\alpha = -I \times \text{angular acceleration} \quad\text{.........(1)},$$
or
$$I\frac{d\alpha}{dt} + F\alpha = 0 \quad\text{....................(2)},$$
or
$$\frac{d\alpha}{dt} = -\frac{F}{I}\alpha.$$

Here rate of diminution of angular velocity α, is proportional to α, so that we have the compound interest law or
$$\alpha = \alpha_0 e^{-\frac{F}{I}t} \quad\text{.........................(3)},$$
where α_0 is the angular velocity at time 0.

Compare this with the case of a fly-wheel stopped **by solid friction**. Let a be the constant solid-frictional torque.

(1) becomes $$a = -I d\alpha/dt,$$

or $$d\alpha/dt + a/I = 0,$$

or $$\alpha = -at/I + \text{a constant},$$

or $$\alpha = \alpha_0 - at/I \quad \text{.....................(4)},$$

where α_0 is the angular velocity when $t = 0$.

Returning to the case of fluid frictional resistance, if M is a varying driving torque applied to a fly-wheel, we have

$$M = F\alpha + I\frac{d\alpha}{dt} \quad \text{.....................(5)}.$$

Notice the analogy here with the following electric circuit law.

Example 8. **Electric Conductor left to itself.**

Ohm's law is for constant currents and is $V = RC$, where R is the resistance of a circuit, C is the current flowing in it; V the voltage. We usually have R in ohms, C in amperes, V in Volts. When the current is not constant, the law becomes

$$\mathbf{V = RC + L\frac{dC}{dt}} \quad \text{.........................(1)},$$

where $\frac{dC}{dt}$ is the rate of increase of amperes per second, and L is called the self-induction of the circuit in Henries. It is evident that L is the voltage retarding the current when the current increases at the rate of one ampere per second.

1. If $V = 0$ in (1)

$$\frac{dC}{dt} = -\frac{R}{L}C,$$

which is the compound interest law.

Consequently $$\mathbf{C = C_0 e^{-\frac{R}{L}t}} \quad \text{......................(2)}.$$

2. If $C = a + be^{-gt}$,

$$\frac{dC}{dt} = -gbe^{-gt},$$

so that from (1) $V = Ra + (Rb - Lgb)\,e^{-gt}$.

Now let $R = Lg$ or $g = \frac{R}{L}$ and we have $V = Ra$, so that the voltage may keep constant although the current alters. Putting in the values we have found, and using V_0 for the constant voltage so that $a = V_0 \div R$, we find

$$C = \frac{V_0}{R} + be^{-\frac{R}{L}t} \quad \text{.......................(3).}$$

If we let $C = 0$ when $t = 0$, then $b = -\frac{V_0}{R}$, and hence we may write

$$C = \frac{V_0}{R}\left(1 - e^{-\frac{R}{L}t}\right) \quad \text{.......................(4).}$$

The curve showing how C increases when a constant voltage is applied to a circuit, ought to be plotted from $t = 0$ for some particular case. Thus plot when $V_0 = 100$, $R = 1$, $L = \cdot 01$. What is the current finally reached?

99. Easy Exercises in the Differentiation and Integration of e^{ax}.

1. Using the formula of Art. 70, find **the radius of curvature** of the curve $y = e^x$, where $x = 0$. Answer: $r = \sqrt{8}$.

2. A point x_1, y_1 is in the curve $y = be^{\frac{x}{a}}$, find the equation to the **tangent** through this point.

Answer: $\dfrac{y - y_1}{x - x_1} = \dfrac{y_1}{a}$.

Find the equation to the **normal** through this point.

Answer: $\dfrac{y - y_1}{x - x_1} = -\dfrac{a}{y_1}$.

Find the length of the **Subnormal.** Answer: $y\dfrac{dy}{dx}$ or y^2/a.

Find the length of the **Subtangent.** Answer: $y \div \dfrac{dy}{dx}$ or a.

3. Find **the radius of curvature** of the **catenary** $y = \frac{c}{2}(e^{x/c} + e^{-x/c})$, at any place. Answer: $r = y^2/c$.

At the vertex when $x = 0$, $r = c$.

4. If $y = Ae^{iax}$ where i stands for $\sqrt{-1}$. Show that if i behaves as an algebraic quantity so that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, &c. then $\mathbf{\frac{d^2y}{dx^2} = -a^2y}$.

5. Find a so that $y = Ae^{ax}$ may be true when

$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 0.$$

Show that there are two values of a and that

$$y = Ae^{-4x} + Be^{-3x}.$$

6. Find the **subtangent and subnormal to the Catenary** $y = \frac{c}{2}(e^{x/c} + e^{-x/c})$, or, as it is sometimes written, $y = c\cosh x/c$.

Answer:

the subtangent is $c\coth\frac{x}{c}$ or $c\,(e^{x/c} + e^{-x/c})/(e^{x/c} - e^{-x/c})$,

the subnormal is $\frac{c}{2}\sinh\frac{2x}{c}$ or $\frac{c}{4}\,(e^{2x/c} - e^{-2x/c})$.

7. The distance PS, fig. 8, being called the length of the tangent, the length of the tangent of the above catenary is

$$\frac{c}{2}\cosh^2\frac{x}{c}\Big/\sinh\frac{x}{c}.$$

The length of PQ may be called the length of the normal, and for the catenary it is $c\cosh^2\frac{x}{c}$ or y^2/c.

8. Find the **length of an arc** of the catenary $y = \frac{c}{2}(e^{\frac{x}{c}} + e^{-\frac{x}{c}})$. The rule is given in Art. 38. Fig. 62 shows the shape of the curve, O being the origin, the distance AO being c. The point P has for its co-ordinates x and y.

Now $\frac{dy}{dx} = \frac{1}{2}(e^{\frac{x}{c}} - e^{-\frac{x}{c}})$. Squaring this and adding to 1 and extracting the square root gives us $\frac{1}{2}(e^{\frac{x}{c}} + e^{-\frac{x}{c}})$. The integral of this is $\frac{c}{2}(e^{\frac{x}{c}} - e^{-\frac{x}{c}})$ which is the length of the arc AP, as it is 0 when $x = 0$. We may write it, $s = c \sinh x/c$.

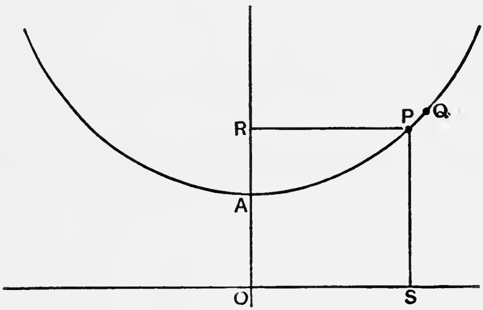

Fig. 62.

9. Find the **area of the catenary** between OA and SP, fig. 62.

$$\text{Area} = \int_0^{OS} \frac{c}{2}(e^{\frac{x}{c}} + e^{-\frac{x}{c}})\,dx,$$

or
$$\frac{c^2}{2}\Big[e^{\frac{x}{c}} - e^{-\frac{x}{c}}\Big]_0^{OS} \text{ or } \frac{c^2}{2}(e^{\frac{OS}{c}} - e^{-\frac{OS}{c}}).$$

Or the area up to any ordinate at x is $c^2 \sinh x/c$.

The Catenary $y = \frac{c}{2}(e^{x/c} + e^{-x/c})$ revolves about the *axis of x*, find the area of the hour-glass-shaped surface generated. See Art. 48.

$$\frac{dy}{dx} = \frac{1}{2}(e^{x/c} - e^{-x/c}) \text{ and } \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{1}{2}(e^{x/c} + e^{-x/c}):$$

$$\text{Area} = \frac{\pi c}{2}\int (e^{x/c} + e^{-x/c})^2 \,.\, dx$$

$$= \frac{\pi c^2}{4}(e^{2x/c} - e^{-2x/c}) + \pi c x$$

between the ordinates at $x = x$ and $x = 0$.

It is curious that the forms of some **volcanoes** are as if their own sections obeyed the compound interest law like an inverted pump rod. The radii of the top and base of such a

volcano being a and b respectively and the vertical height h, **find the volume.** See Art. 46. Taking the axis of the volcano as the axis of x, the curve $y = be^{cx}$ revolving round this axis will produce the outline of the mountain if $c = \frac{1}{h}\log\frac{a}{b}$.

$$\text{The volume is } \pi\int_0^h b^2 \,.\, e^{2cx} \,.\, dx = \frac{\pi b^2}{2c}\left[{}_0^h\, e^{2cx}\right]$$
$$= \frac{\pi b^2}{2c}(e^{2ch} - 1).$$

Now $a = be^{ch}$, so that our answer is $\frac{\pi}{2c}(a^2 - b^2)$.

100. Harmonic Functions.

Students ought to have already plotted **sine curves** like $y = a\sin(bx + e)$...(1) on squared paper and to have figured out for themselves the signification of a, b and e. It ought to be unnecessary here to speak of them. Draw the curve again. Why is it sometimes called a cosine curve? [Suppose e to be $\frac{\pi}{2}$ or 90°.] Note that however great x may be, the sine of $(bx + e)$ can never exceed 1 and never be less than -1. The student knows of course that $\sin 0 = 0$, $\sin\frac{\pi}{4}$ (or 45°) = ·707, $\sin\frac{\pi}{2}$ (or 90°) = 1, $\sin\frac{3\pi}{4}$ (or 135°) = ·707, $\sin\pi$ (or 180°) = 0, $\sin\frac{5\pi}{4}$ (or 225°) = $-$ ·707, $\sin\frac{3\pi}{2}$ (or 270°) = -1, $\sin\frac{7\pi}{4}$ (or 315°) = $-$·707, $\sin 2\pi$ (or 360°) = 0 again and, thereafter, $\sin\theta = \sin(\theta - 2\pi)$. Even these numbers ought almost to be enough to let the wavy nature of the curve be seen. Now as a sine can never exceed 1, the greatest and least values of y are a and $-a$. Hence a is called the **amplitude** of the curve or of the function.

When $x = 0$, $y = a\sin e$. This gives us the signification of e. Another way of putting this is to say that when bx was $= -e$ or $x = -\frac{e}{b}$, y was 0. When x indicates time or when bx is the angle passed through by a crank or an eccentric, e gets several names; Valve-motion engineers call

it the *advance* of the valve; Electrical engineers call it the *lead* or (if it is negative) the *lag*.

Observe that when $bx = 2\pi$ we have everything exactly the same as when x was 0, so that we are in the habit of calling $\frac{2\pi}{b}$ the periodic value of x.

Besides the method given in Art. 9, I advise the student to draw the curve by the following method. A little knowledge of elementary trigonometry will show that it must give exactly the same result. It is just what is done in drawing the elevation of a spiral line (as of a screw thread) in the drawing office. Draw a straight line OM. Describe

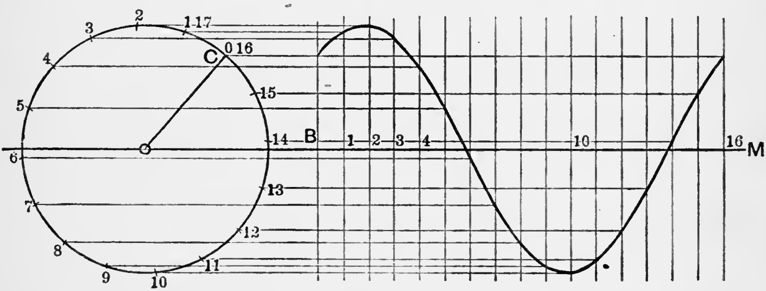

Fig. 63.

a circle about O with a as radius. Set off the angle BOC equal to e. Divide the circumference of the circle into any number of equal parts numbering the points of division 0, 1, 2, 3, &c. We may call the points 16, 17, 18, &c., or 32, 33, 34, &c., when we have gone once, twice or more times round. Set off any equal distances from B towards M on the straight line, and number the points 0, 1, 2, &c. Now project vertically and horizontally and so get points on the curve. The distance BM represents to some scale or other the periodic value of x or $2\pi/b$.

If OC is imagined to be a crank rotating uniformly against the hands of a watch in the vertical plane of the paper, y in (1) means the distance of C above OM, bx means the angle that OC makes at any time with the position OM, and if x means time, then b is the angular velocity of the crank and $2\pi/b$ means the time of one revolution of the crank

or the periodic time of the motion. y is the displacement at any instant, from its mid position, of a slider worked vertically from C by an infinitely long connecting rod.

A simple harmonic motion may be defined as one which is represented by $s = a \sin(bt + e)$, where s is the distance from a mid position, a is the amplitude, e the lead or lag or advance, and b is $2\pi/T$ or $2\pi f$ where T is the periodic time or f is the frequency. Or it may be defined as the motion of a point rotating uniformly in a circle, projected upon a diameter of the circle (much as we see the orbits of Jupiter's satellites, edge on to us), or the motion of a slider worked from a uniformly rotating crank-pin by means of an infinitely long connecting rod. And it will be seen later, that it is the sort of motion which a body gets when the force acting upon it is proportional to the distance of the body from a position of equilibrium, as in the up and down motion of a mass hanging at the end of a spring, or the bob of a pendulum when its swings are small. It is the simplest kind of vibrational motion of bodies. Many pairs of quantities are connected by such a sine law, as well as space and time, and we discuss simple harmonic motion less, I think, for its own sake, than because it is analogous to so many other phenomena. Now let it be well remembered although not yet proved that if

$$\mathbf{y = a \sin(bx + c)} \text{ then } \mathbf{\frac{dy}{dx} = ab \cos(bx + c)}$$

and
$$\mathbf{\int y \, . \, dx = -\frac{a}{b} \cos(bx + c).}$$

101. When $c = 0$ and $b = 1$ and $a = 1$; that is when

$$y = \sin x \quad \text{.........................(1)},$$

let us find the differential coefficient.

As before, let x be increased to $x + \delta x$ and find $y + \delta y$,

$$y + \delta y = \sin(x + \delta x) \quad \text{.... ....(2)}.$$

Subtract (1) from (2) and we find

$$\delta y = \sin(x + \delta x) - \sin x,$$

or
$$2 \cos(x + \tfrac{1}{2}\delta x) \sin \tfrac{1}{2}\delta x. \quad \text{(See Art. 3.)}$$

Hence
$$\frac{\delta y}{\delta x} = \cos(x + \tfrac{1}{2}\delta x) \frac{\sin \tfrac{1}{2}\delta x}{\tfrac{1}{2}\delta x} \quad \text{..............(3)}.$$

It is easy to see by drawing a small angle α and recollecting what $\sin \alpha$ and α are, to find the value of $\sin \alpha \div \alpha$ as α gets smaller and smaller. Thus in the figure, let POA be the angle. The arc PA divided by OP is α, the angle in radians. And the perpendicular PB divided by OP is the sine of the angle. Hence

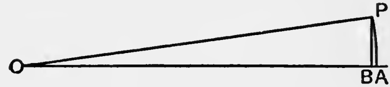

Fig. 64.

$\frac{\sin \alpha}{\alpha} = \frac{PB}{PA}$ and it is evident that this is more and more nearly 1 as α gets smaller and smaller. **In fact we may take the ratios of α, $\sin \alpha$ and $\tan \alpha$ to one another to be 1, more and more nearly, as α gets smaller and smaller.** If we look upon $\frac{1}{2}\delta x$ as α in the above expression, we see that in the limit, (3) becomes

$$\frac{dy}{dx} = \cos x \text{ if } y = \sin x.*$$

* The proof of the more general case is of exactly the same kind. Here it is:

$$y = a \sin (bx + c),$$

$$y + \delta y = a \sin \{b (x + \delta x) + c\},$$

$$\delta y = 2a \cos (bx + c + \tfrac{1}{2} b \, . \, \delta x) \sin (\tfrac{1}{2} b \, . \, \delta x). \quad \text{See Art. 3.}$$

$$\frac{\delta y}{\delta x} = ab \, . \cos (bx + c + \tfrac{1}{2} b \, . \, \delta x) \frac{\sin (\frac{1}{2} b \, . \, \delta x)}{\frac{1}{2} b \, . \, \delta x}.$$

Now make δx smaller and smaller and we have

$$\frac{dy}{dx} = ab \cos (bx + c) \text{ and hence } \int a \cos (bx + c) \, . \, dx = \frac{a}{b} \sin (bx + c).$$

Again, to take another case:—

If $y = a \cos (bx + e)$, this is the same as

$$y = a \sin \left(bx + e + \frac{\pi}{2} \right) = a \sin (bx + c), \text{ say.}$$

Hence
$$\frac{dy}{dx} = ab \cos (bx + c)$$

$$= ab \cos \left(bx + e + \frac{\pi}{2} \right)$$

$$\frac{dy}{dx} = -ab \sin (bx + e).$$

Hence
$$\int a \sin (bx + c) \, . \, dx = -\frac{a}{b} \cos (bx + c).$$

And hence $$\int \cos x \,.\, dx = \sin x.$$

102. Now it is not enough to prove a thing like this, it must be known. Therefore the student ought to take a book of Mathematical Tables and illustrate it. It is unfortunate that such books are arranged either for the use of very ignorant or else for very learned persons and so it is not quite easy to convert radians into degrees or *vice versa.* Do not forget that in $\sin x$ or $\cos x$ we mean x to be in radians. Make out such a little bit of table as this, which is taken at random.

Angle in degrees	x or angle in radians	$y = \sin x$	δy	$\delta y \div \delta x$	Average $\frac{\delta y}{\delta x}$
40	·6981	·6427876			
			·0132714	·7583	
41	·7156	·6560590			·7547
			·0130716	·7512	
42	·7330	·6691306			

If it is remembered that $\delta y \div \delta x$ in each case is really the *average* value of $\frac{dy}{dx}$ for one degree or ·01745 of a radian, it will be seen why it is not exactly equal to the cosine of x. Has the student looked for himself, to see if ·7547 is really nearly equal to $\cos 41°$?

103. It is easy to show in exactly the same way that if $\mathbf{y = \cos x}$, $\mathbf{\frac{dy}{dx} = -\sin x}$ and $\mathbf{\int \sin x \,.\, dx = -\cos x}$. The $-$ sign is troublesome to remember. Here is an illustration:

Angle in degrees	x or angle in radians	$y = \cos x$	δy negative	$\frac{\delta y}{\delta x}$	Average $\frac{\delta y}{\delta x}$
20	·3491	·9396926			
			·0061122	$-$·3513	
21	·3665	·9335804			$-$ ·3584
			·0063965	$-$·3656	
22	·3840	·9271839			

Notice that y diminishes as x increases. Notice that $\sin 21^\circ$ or $\sin(\cdot 3665) = \cdot 3584$.

104. Here is another illustration of the fact that the differential coefficient of $\sin x$ is $\cos x$. Let AOP, fig. 65, be θ. Let AOQ be $\theta + \delta\theta$. Let PQ be a short arc drawn with O as centre. Let $OP = OQ = 1$. PR is perpendicular to QB.

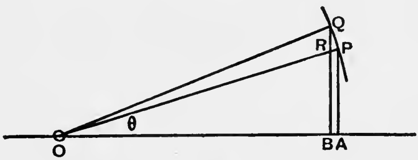

Fig. 65.

Then $AP = y = \sin\theta$, $BQ = \sin(\theta + \delta\theta) = y + \delta y$, $QP = \delta\theta$ and $RQ = \delta y$. Now the length of the arc PQ becomes more and more nearly the length of a straight line between P and Q as $\delta\theta$ is made smaller and smaller.

Thus $\frac{RQ}{QP}$ or $\frac{\delta y}{\delta\theta}$ is more and more nearly equal to $\cos PQR$ or $\cos\theta$.

In the limit $\mathbf{\frac{dy}{d\theta} = \cos\theta}$ if $\mathbf{y = \sin\theta}$.

Similarly if $\mathbf{z = \cos\theta} = OA$, $\delta z = -BA = -RP$ and

$$\frac{dz}{d\theta} = -\frac{RP}{QP} = -\sin\theta.$$

Illustrations like these are however of most value when a student invents them for himself. Any way of making the fundamental ideas familiar to oneself is valuable. But it is a great mistake for the author of a book to give too many illustrations. He is apt to give prominence to those illustrations which he himself discovered and which were therefore invaluable in his own education.

105. Observe that if $y = A\sin ax + B\cos ax$,

$$\frac{d^2y}{dx^2} = -a^2y, \text{ and } \frac{d^4y}{dx^4} = a^4y.$$

Compare this with the fact that if $y = e^{ax}$, $\frac{d^2y}{dx^2} = a^2y$, $\frac{d^4y}{dx^4} = a^4y$. In

the higher applications of Mathematics to Engineering this resemblance and difference between the two functions e^{ax} and $\sin ax$ become important. Note that if i stands for $\sqrt{-1}$ so that $i^2=-1$, $i^4=1$, &c.

Then if $y=e^{iax}$, $\frac{d^2y}{dx^2}=-a^2y$, $\frac{d^4y}{dx^4}=a^4y$ just as with the sine function. Compare Art. 99.

106. *Exercise.* Men who have proved Demoivre's theorem in Trigonometry (the proof is easy; the proofs of all mathematical rules which are of use to the engineer are easy; difficult proofs are only useful in academic exercise work) say that for all algebraic purposes, $\cos ax=\frac{1}{2}(e^{iax}+e^{-iax})$ and $\sin ax=\frac{1}{2i}(e^{iax}-e^{-iax})$. If this is so, prove our fundamental propositions.

107. *Example.* A plane **electric circuit** of area A sq. cm. closed on itself, can rotate with uniform angular velocity about an axis which is at right angles to the field, in a uniform magnetic field H. H is supposed given in C.G.S. units; measuring the angle θ as the angle passed through from the position when there is maximum induction HA through the circuit; in the position θ, the induction through the circuit is evidently $A \cdot H \cdot \cos\theta$. If the angle θ has been turned through in the time t with the angular velocity q radians per second, then $\theta=qt$. So that the induction $I=AH\cos qt$. The rate of increase of this per second is $-AqH\sin qt$, and this is the electromotive force in each turn of wire. If there are n turns, the total voltage is $-nAqH\sin qt$ in C.G.S. units; if we want it in commercial units **the voltage is**

$$-\mathbf{nAqH\,10^{-8}\sin qt} \text{ volts},$$

being a simple harmonic function of the time. Note that the term voltage is now being employed for the line integral of electromotive force even when the volt is not the unit used.

Example. The **coil of an alternator** passes through a field such that the induction through the coil is

$$I=A_0+A_1\sin(\theta+\epsilon_1)+A_r\sin(r\theta+\epsilon_r),$$

where θ is the angle passed through by the coil. If q is the

relative angular velocity of the coil and field, $\theta = qt$. If there are n turns of wire on the coil, then the voltage is $n\frac{dI}{dt}$, or

$$nq\{A_1 \cos(qt + \epsilon_1) + A_r r \cos(rqt + \epsilon_r)\}.$$

So we see that **irregularities** of r times the frequency **in the field** are relatively multiplied or **magnified** in the electromotive force.

108. In **Bifilar Suspension**, if W is the weight of the suspended mass, a and b the distances between the threads below and above, h the vertical height of the threads; if the difference in vertical component of tension is n times the total weight W, and θ is the angle turned through in azimuth, the momental resistance offered to further turning is

$$\tfrac{1}{4}(1 - n^2)\, W\frac{ab}{h}\sin\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Note that to make the arrangement more 'sensitive' it is only necessary to let more of the weight be carried by one of the threads than the other.

The momental resistance offered to turning by a body which is θ from its position of equilibrium, is often proportional to $\sin\theta$. Thus if W is the weight of a compound **pendulum** and OG is the distance from the point of support to its centre of gravity, $W.OG.\sin\theta$ is the moment with which the body tends to return to its position of equilibrium. If M is the magnetic moment of a **magnet** displaced θ from equilibrium in a field of strength H, then $HM\sin\theta$ is the moment with which it tends to return to its position of equilibrium. A body constrained by the **torsion** of a wire or a strip has a return moment proportional to θ. When angular changes are small we often treat $\sin\theta$ as if it were equal to θ. Sometimes a body may have various kinds of constraint at the same time. Thus the needle of a quadrant electrometer has bifilar suspension, and there is also an **electrical constraint** introduced by bad design and construction which may perhaps be like $a\theta + b\theta^2$. If the threads are stiff, their own torsional stiffness introduces a term proportional to θ which we did not include in (1). Sometimes the constraint is introduced by connecting a little magnetic needle rigidly

with the electrometer needle, and this introduces a term proportional to $\sin\theta$. In some instruments where the moving body is soft iron the constraint is nearly proportional to $\sin 2\theta$. Now if the resisting moment is M and a body is turned through the angle $\delta\theta$, the work done is $M.\delta\theta$. Hence the work done in turning a body from the position θ_1 to the position θ_2, where θ_2 is greater than θ_1, is $\int_{\theta_1}^{\theta_2} M.d\theta$.

Example. The momental resistance offered by a body to turning is $a\sin\theta$ where θ is the angle turned through, what work is done in turning the body from θ_1 to θ_2?

Answer, $\int_{\theta_1}^{\theta_2} a.\sin\theta.d\theta = -a(\cos\theta_2 - \cos\theta_1) = a(\cos\theta_1 - \cos\theta_2)$.

Example. The resistance of a body to turning is partly a constant torque a due to friction, partly a term $b\theta + c\theta^2$, partly a term $e\sin\theta$; what is the work done in turning from $\theta = 0$ to any angle?

M the torque $= a + b\theta + c\theta^2 + e\sin\theta = f(\theta)$ say,

V the work done

$$= a\theta + \tfrac{1}{2}b\theta^2 + \tfrac{1}{3}c\theta^3 + e(1 - \cos\theta) = F(\theta) \text{ say.}$$

This is called the **potential energy** of the body in the position θ.

The **kinetic energy** in a rotating body is $\frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2$ where I is the body's moment of inertia about its axis. When a body is at θ its total energy E is $\frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2 + F(\theta)$.

If the total energy remains constant and a body in the position θ is moving in the direction in which θ increases, and no force acts upon it except its constraint, it will continue to move to the position θ_1 such that

$$\tfrac{1}{2}I\left(\frac{d\theta}{dt}\right)^2 + F(\theta) = F(\theta_1).$$

So that when the form of $F(\theta)$ is known, θ_1 can be calculated, if we know the kinetic energy at θ.

Thus let $M = e \sin\theta$, so that $F(\theta) = e(1 - \cos\theta)$, then

$$\tfrac{1}{2}I\left(\frac{d\theta}{dt}\right)^2 + e(1 - \cos\theta) = e(1 - \cos\theta_1),$$

from which θ_1 **the extreme swing can be calculated.**

Exercise. Show that if the righting moment of a ship is proportional to $\sin 4\theta$ where θ is the heeling angle, and if a wind whose momental effect would maintain a steady inclination of $11\frac{2}{3}$ degrees suddenly sends the ship from rest at $\theta = 0$ and remains acting, and if we may neglect friction, the ship will heel beyond $33\frac{1}{3}$ degrees and will go right over. Discuss the effect of friction.

A body is in the extreme position θ_1, **what will be its kinetic energy** when passing through the position of equilibrium? Answer, $F(\theta_1)$.

Thus let $M = b\theta + c\theta^2 + e\sin\theta$.

Calculate α, the angular velocity at $\theta = 0$, if its extreme swing is 45°. Here

$$\theta_1 = \frac{\pi}{4} \text{ and } F(\theta) = \tfrac{1}{2}b\theta^2 + \tfrac{1}{3}c\theta^3 + e(1 - \cos\theta),$$

$$\tfrac{1}{2}I\alpha^2 = \tfrac{1}{2}b\left(\frac{\pi}{4}\right)^2 + \tfrac{1}{3}c\left(\frac{\pi}{4}\right)^3 + e\left(1 - \frac{1}{\sqrt{2}}\right),$$

from which we may calculate α.

Problem. Suppose we desire to have the potential energy following the law

$$V = F(\theta) = a\theta^{\frac{3}{2}} + b\theta^3 + c\epsilon^{m\theta} + h\sin 2\theta,$$

and we wish to know the necessary law of constraint, we see at once that as $M = \dfrac{dV}{d\theta}$,

$$M = \tfrac{3}{2}a\theta^{\frac{1}{2}} + 3b\theta^2 + mc\epsilon^{m\theta} + 2h\cos 2\theta.$$

Problem. A body in the position θ_0 moving with the angular velocity α, in the direction of increasing θ, **has a momental impulse** m in the direction of increasing θ suddenly **given to it; how far will it swing?**

The moment of momentum was $I\alpha$, it is now $I\alpha + m$ and if α' is its new angular velocity $\alpha' = \alpha + \dfrac{m}{I}$.

The body is then in the position θ_0 with the kinetic energy $\frac{1}{2}I\left(\alpha+\frac{m}{I}\right)^2$ and the potential energy $F(\theta_0)$, and the sum of these equated to $F(\theta_1)$ enables θ_1 to be calculated.

The student will easily see that the general equation of angular motion of the constrained body is

$$\frac{d^2\theta}{dt^2}+\frac{1}{I}f(\theta)=0\,;$$

$f(\theta)$ may include a term involving friction.

109. Every one of the following exercises must be worked carefully by students; the answers are of great practical use but more particularly to **Electrical Engineers.** In working them out it is necessary to recollect the trigonometrical relations

$$\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta,$$
$$2\sin\theta\,.\cos\phi = \sin(\theta+\phi)+\sin(\theta-\phi),$$
$$2\cos\theta\,.\cos\phi = \cos(\theta+\phi)+\cos(\theta-\phi),$$
$$2\sin\theta\,.\sin\phi = \cos(\theta-\phi)-\cos(\theta+\phi).$$

Such exercises are not merely valuable in illustrating the calculus; they give an acquaintance with trigonometrical expressions which is of great general importance to the engineer.

The average value of $f(x)$ from $x=x_1$ to $x=x_2$ is evidently the area $\int_{x_1}^{x_2} f(x)\,.\,dx$ divided by x_2-x_1.

Every exercise from 6 to 20 and also 23 ought to be illustrated graphically by students. Good hand sketches of the curves whose ordinates are multiplied together and of the resulting curves will give sufficiently accurate illustrations.

1. $\int \sin^2 ax dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$.

2. $\int \cos^2 ax dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$.

3. $\int \sin ax\,.\cos bx\,.\,dx = -\frac{\cos(a+b)x}{2(a+b)} - \frac{\cos(a-b)x}{2(a-b)}$.

4. $\int \sin ax \,.\, \sin bx \,.\, dx = \frac{\sin (a-b)x}{2(a-b)} - \frac{\sin (a+b)x}{2(a+b)}.$

5. $\int \cos ax \,.\, \cos bx \,.\, dx = \frac{\sin (a+b)x}{2(a+b)} + \frac{\sin (a-b)x}{2(a-b)}.$

6. The area of a sine curve for a whole period is 0.

$$\int_0^{2\pi} \sin x \,.\, dx = -\left[\cos x\right]_0^{2\pi} = -(1-1) = 0.$$

7. Find the area of the positive part of a sine curve, that is

$$\int_0^{\pi} \sin x \,.\, dx = -\left[\cos x\right]_0^{\pi} = -(-1-1) = 2.$$

Since the length of base of this part of the curve is π, the average height of it is $\frac{2}{\pi}$. Its greatest height, or amplitude, is 1.

8. The area of $y = a + b \sin x$ from 0 to 2π is $2\pi a$ and the average height of the curve is a.

9. Find the average value of $\sin^2 x$ from $x=0$ to $x=2\pi$.

As $\cos 2x = 1 - 2\sin^2 x$, $\sin^2 x = \frac{1}{2}(1-\cos 2x)$. The integral of this is $\frac{1}{2}x - \frac{1}{4}\sin 2x$, and putting in the limits, the area is $(\frac{1}{2}\, 2\pi - \frac{1}{4}\sin 4\pi - 0 + \frac{1}{4}\sin 0) = \pi$. The average height is the area $\div\, 2\pi$, and hence it is $\frac{1}{2}$.

10. The average value of $\cos^2 x$ from $x=0$ to $x=2\pi$ is $\frac{1}{2}$.

In the following exercises s and r are supposed to be whole numbers and unequal:

11. The average value of $a\sin^2(sqt+e)$ from $t=0$ to $t=T$ is $\frac{a}{2}$, s being a whole number and $q = 2\pi/T$. T is the periodic time.

12. The average value of $a\cos^2(sqt+e)$ from $t=0$ to $t=T$ is $\frac{a}{2}$.

13. $\int_0^T \cos sqt \,.\, \sin sqt \,.\, dt = 0.$

14. $\int_0^T \sin sqt \,.\, \sin rqt \,.\, dt = 0.$

15. $\int_0^T \cos sqt \,.\, \cos rqt \,.\, dt = 0.$

16. $\int_0^T \sin sqt \,.\, \cos rqt \,.\, dt = 0.$

17. The average value of $\sin^2 sqt$ from 0 to $\frac{1}{2}T$ is $\frac{1}{2}$.

18. The average value of $\cos^2 sqt$ from 0 to $\frac{1}{2}T$ is $\frac{1}{2}$.

19. $\int_0^{\frac{1}{2}T} \sin sqt \,.\, \sin rqt \,.\, dt = 0.$

20. $\int_0^{\frac{1}{2}T} \cos sqt \,.\, \cos rqt \,.\, dt = 0.$

21. Find $\int \sin x \,.\, \sin (x + e) \,.\, dx.$

Here, $\sin (x + e) = \sin x \cos e + \cos x \,.\, \sin e.$

Hence we must integrate $\sin^2 x \,.\, \cos e + \sin x \,.\, \cos x \,.\, \sin e,$

$$\int \sin^2 x \,.\, dx = \frac{x}{2} - \frac{\sin 2x}{4},$$

$$\int \sin x \,.\, \cos x \,.\, dx = \tfrac{1}{2} \int \sin 2x \,.\, dx = -\tfrac{1}{4} \cos 2x,$$

and hence our integral is

$$\left(\frac{x}{2} - \frac{\sin 2x}{4}\right) \cos e - \tfrac{1}{4} \cos 2x \,.\, \sin e.$$

22. Prove that $\int \sin qt \,.\, \sin (qt + e) \,.\, dt$

$$= \left(\frac{t}{2} - \frac{\sin 2qt}{4q}\right) \cos e - \frac{1}{4q} \cos 2qt \,.\, \sin e.$$

23. Prove that the average value of $\sin qt \,.\, \sin (qt \pm e)$ or of $\sin (qt + a) \sin (qt + a \pm e)$ for the whole periodic time T $\left(\text{if } q = \frac{2\pi}{T}\right)$ is $\frac{1}{2} \cos e$.

This becomes evident when we notice (calling $qt + a = \phi$),

$$\sin\phi \,.\, \sin(\phi \pm e) = \sin\phi\,(\sin\phi\cos e \pm \cos\phi \,.\, \sin e)$$
$$= \sin^2\phi \,.\, \cos e \pm \sin\phi \,.\, \cos\phi \,.\, \sin e.$$

Now the average value of $\sin^2\phi$ for a whole period is $\frac{1}{2}$, and the average value of $\sin\phi \,.\, \cos\phi$ is 0.

By making $a = \frac{\pi}{2}$ in the above we see that the average value of $\cos qt \,.\, \cos(qt \pm e)$ is $\frac{1}{2}\cos e$, **or the average value for a whole period of the product of two sine functions of the time, of the same period, each of amplitude 1, is half the cosine of the angular lag of either behind the other.**

24. Referring to Art. 106, take

$$\cos a\theta = \tfrac{1}{2}\left(e^{ia\theta} + e^{-ia\theta}\right),$$
$$\sin a\theta = \frac{1}{2i}\left(e^{ia\theta} - e^{-ia\theta}\right),$$

or take
$$e^{ia\theta} = \cos a\theta + i\sin a\theta,$$
$$e^{-ia\theta} = \cos a\theta - i\sin a\theta,$$

and find
$$\int e^{b\theta}\cos a\theta \,.\, d\theta.$$

This becomes $\frac{1}{2}\int\left(e^{(b+ai)\theta} + e^{(b-ai)\theta}\right)d\theta$

$$= \tfrac{1}{2}\left\{\frac{1}{b+ai}e^{(b+ai)\theta} + \frac{1}{b-ai}e^{(b-ai)\theta}\right\}$$
$$= \tfrac{1}{2}e^{b\theta}\left\{\frac{1}{b+ai}e^{ai\theta} + \frac{1}{b-ai}e^{-ai\theta}\right\},$$

and on substituting the above values it becomes

$$\int e^{b\theta}\cos a\theta \,.\, d\theta = \frac{1}{a^2+b^2}e^{b\theta}(b\cos a\theta + a\sin a\theta)\ \ldots(1).$$

Similarly we have

$$\int e^{b\theta}\sin a\theta d\theta = \frac{1}{a^2+b^2}e^{b\theta}(b\sin a\theta - a\cos a\theta)\ \ldots\ldots(2).$$

110. Notes on Harmonic Functions. In the following collection of notes the student will find a certain amount of repetition of statements already made.

111. A function $x = a \sin qt$ is analogous to the straight line motion of a slider driven from a **crank of length** a (rotating with the **angular velocity** q radians per second) by an infinitely long connecting rod. x is the distance of the slider from the middle of its path at the time t. At the zero of time, $x = 0$ and the crank is at right angles to its position of dead point, $\mathbf{q = 2\pi f = \frac{2\pi}{T}}$, if T is the **periodic time**, or if f is the **frequency** or number of revolutions of the crank per second, taking 1 second as the unit of time.

112. A function $x = a \sin (qt + \epsilon)$ is just the same, except that the crank is the angle ϵ **radians** (one radian is 57·2957 degrees) **in advance** of the former position; that is, at time 0 the slider is the distance $a \sin \epsilon$ past its mid-position*.

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* The student is here again referred to § 10, and it is assumed that he has drawn a curve to represent

$$x = b\epsilon^{-at} \sin (qt + \epsilon) \text{ ....(1).}$$

Imagine a crank to rotate uniformly with the angular velocity q, and to drive a slider, but imagine the crank to get shorter as time goes on, its length at any time being $a\epsilon^{-bt}$.

Another way of thinking of this motion is:—

Imagine a point P to move with constant *angular* velocity round O,

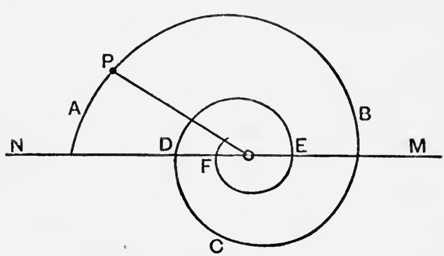

Fig. 66.

keeping in the equiangular spiral path $APBCDEF$; the motion in question is the motion of P projected upon the straight line MON and what we have called the logarithmic decrement is $\pi \cot \alpha$ if α is the angle of the spiral,

113. A function $x = a \sin(qt + \epsilon) + a' \sin(qt + \epsilon')$ is the same as $X = A \sin(qt + E)$; that is, the **sum of two crank motions can be given by a single crank of proper length and proper advance.** Show on a drawing the positions of the first two when $t = 0$, that is, set off

$$YOP = \epsilon \text{ and } OP = a,$$
$$YOQ = \epsilon' \text{ and } OQ = a'.$$

Complete the parallelogram $OPRQ$ and draw the diagonal OR, then the single crank $OR = A$, with angle of advance $YOR = E$, would give to a slider the sum of the motions which OP and OQ would separately give. The geometric proof of this is very easy. Imagine the slider to have a vertical motion. Draw OQ, OR and OP in their relative positions at any time, then project P, R and Q upon OX. The crank OP would cause the slider to be OP' above its mid-position at this instant, the crank OQ would cause the slider to be OQ' above its mid-position, the crank OR would cause the slider to be OR' above its mid-position at the same instant; observe that OR' is *always* equal to the algebraic sum of OP' and OQ'.

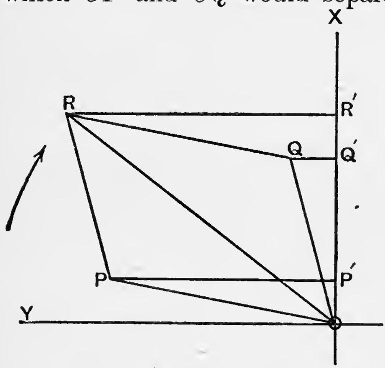

Fig. 67.

We may put it thus:—"The S.H.M. which the crank OP would give, + the S.H.M. which OQ would give, is equal to the S.H.M. which OR would give." Similarly "the S.H.M. which OR would give, − the S.H.M. which OP would give, is equal to the S.H.M. which OQ would give." We sometimes say:—the crank OR is the sum of the two cranks OP and OQ. Cranks are added therefore and subtracted just like vectors.

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that is, the constant acute angle which OP everywhere makes with the curve, or $\pi \cot \alpha = aT/2$ and $q = 2\pi/T$, so that $\cot \alpha = a/q$. If fig. 66 is to agree with fig. 67 in all respects NM being vertical and P is the position at time 0, then $\epsilon = \text{angle } NOP - \pi/2$.

114. These propositions are of great importance in dealing with valve motions and other mechanisms. They are of so much importance to electrical engineers, that many practical men say, "let the crank OP represent the current." They mean, "there is a current which alters with time according to the law $C = a \sin(qt + \epsilon)$, its magnitude is analogous to the displacement of a slider worked vertically by the crank OP whose length is a and whose angular velocity is q and OP is its position when $t = 0$." †

115. Inasmuch as the function $x = a \cos qt$ is just the same as $a \sin\left(qt + \frac{\pi}{2}\right)$, it represents the motion due to a crank of length a whose angle of advance is 90°. At any time t the **velocity** of a slider whose motion is

$$x = a \sin(qt + \epsilon),$$

is

$$v = aq \cos(qt + \epsilon) = \frac{dx}{dt} \text{ or } \dot{x}$$

$$= aq \sin\left(qt + \epsilon + \frac{\pi}{2}\right),$$

that is, it can be represented by the actual position at any instant of a slider worked by a crank of length representing aq, this new crank being 90° in advance of the old one. The **acceleration** or $\frac{d^2x}{dt^2}$ or $\frac{dv}{dt}$ or $\dot{v}$ is shown at any instant by a crank of length aq^2 placed 90° in advance of the v crank, or 180° in advance of the x crank, for

$$\text{Accel.} = -aq^2 \sin(qt + \epsilon)$$

$$= aq^2 \sin(qt + \epsilon + \pi).$$

The characteristic property of S.H. motion is that, numerically, the acceleration is q^2 or $4\pi^2 f^2$ times the displacement, f being the frequency.

If anything follows the law $a \sin(qt + \epsilon)$, it is analogous to the motion of a slider, and we often say that it is represented by the crank OP; **its rate of increase with time** is analogous to the velocity of the slider, and we say that it is represented by a crank of length aq placed 90° in advance of the first. In fact, on a S.H. function, the operator d/dt multiplies by q and gives an advance of a right angle.

116. Sometimes instead of stating that a function is **A sin (qt + ε)** we state that it is **a sin qt + b cos qt.**

Evidently this is the same statement, if $a^2+b^2=A^2$ and if $\tan\epsilon=\dfrac{b}{a}$.

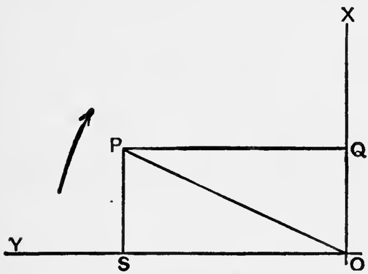

Fig. 68.

It is easy to prove this trigonometrically, and graphically in fig. 68. Let

$$OS=a,\ OQ=b.$$

The crank OP is the sum of OS and OQ, and $\tan\epsilon$ or

$$\tan YOP=\frac{b}{a}.$$

117. We have already in Art. 100 indicated an easy graphical method of drawing the curve

$$x=a\sin(qt+\epsilon),$$

where x and t are the ordinate and abscissa.

Much information is to be gained by drawing the two of the same periodic time,

$$x=a\sin(qt+\epsilon) \text{ and } x=a'\sin(qt+\epsilon'),$$

and adding their ordinates together. This will illustrate 113.

118. If the voltage in an **Electric Circuit** is V volts, the current C ampères, the resistance R ohms, the self-induction L Henries, then if t is time in seconds,

$$V=RC+L\frac{dC}{dt}\text{.........................(1).}$$

Now if $C=C_0\sin qt$,

$$\frac{dC}{dt}=C_0q\cos qt,$$

so that $\qquad V=RC_0\sin qt+LC_0q\,.\cos qt,$

and by Art. 116 this is

$$V=C_0\sqrt{R^2+L^2q^2}\sin(qt+\epsilon)\text{ ....(2);}$$

$\sqrt{R^2+L^2q^2}$ is called the impedance;

$$\tan\epsilon = \frac{Lq}{R} = \frac{2\pi Lf}{R}, \text{ if } f \text{ is frequency;}$$

ϵ is *lag* of current behind voltage.

Hence again if $\quad \mathbf{V = V_0 \sin qt}$ ....(3),

then $\quad \mathbf{C = \dfrac{V_0}{\sqrt{R^2+L^2q^2}} \sin\left(qt - \tan^{-1}\dfrac{Lq}{R}\right)}$ † ....(4).

Notice that if V is given as in (3) the complete answer for C includes an **evanescent term** due to the starting conditions see Arts. 98, 147, but (4) assumes that the simple harmonic V has been established for a long time. In practical electrical working, a small fraction of a second is long enough to destroy the evanescent term.

119. We may write the characteristic property of a simple harmonic motion as

$$\mathbf{\frac{d^2x}{dt^2} + q^2x = 0} \text{ ....(1),}$$

(compare Arts. 26 and 108) and if (1) is given us we know that it means

$$x = a\sin qt + b\cos qt \quad \text{or} \quad x = A\sin(qt+\epsilon) \text{......(2),}$$

where A and ϵ, or a and b are any arbitrary constants.

Example. A **body** whose weight is W lb. **has a simple harmonic motion** of amplitude a feet (that is, the stroke is $2a$ feet) and has a frequency f per second, **what forces** give to the mass this motion?

If x feet is the displacement of the body from mid-position at any instant, we may take the motion to be

$$x = a\sin qt \quad \text{or} \quad a\sin 2\pi f\,.\,t,$$

and the numerical value of the acceleration at any instant is $4\pi^2f^2x$ and the force drawing the body to its mid-position is in pounds $4\pi^2f^2xW \div 32{\cdot}2$, as mass in engineer's units is weight in pounds in London $\div 32{\cdot}2$, and force is acceleration $\times$ mass.

120. If the connecting rod of a **steam or gas engine** were long enough, and we take W to be the weight of piston and rod, the above is nearly the force which must be exerted by the cross-head when the atmosphere is admitted to both sides of the piston. Observe that it is 0 when x is 0 and is proportional to x, being greatest at the ends of the stroke. Make a diagram showing how much this force is at every point of the stroke, and carefully note that it is always acting *towards* the middle point.

Now if the student has the indicator diagrams of an engine (both sides of piston), he can first draw a diagram showing at every point of the stroke the force of the steam on the piston, and he can combine this with the above diagram to show the actual force on the cross-head. Note that steam pressure is so much per square inch, whereas the other is the total force. If the student carries out this work by himself it is ten times better than having it explained.

Since the acceleration is proportional to the *square* of the frequency, vibrations of engines are much more serious than they used to be, when speeds were slower.

121. As we have been considering the motion of the piston of a steam engine on the assumption that the connecting rod is infinitely long, we shall now study the effect of **shortness of connecting rod.**

In Art. 11, we found s the distance of the piston from the end of its stroke when the crank made an angle θ with its dead point. Now let x be the distance of the piston to the right of the middle of its stroke in fig. 3, so that our x is the old s minus r, where r is the length of the crank.

Let the crank go round uniformly at q radians per second.

Again, let t be the time since the crank was at right angles to its dead point position, so that $\theta - \frac{\pi}{2} = qt$, and we find

$$x = -r\cos\theta + l\left\{1 - \sqrt{1 - \frac{r^2}{l^2}\sin^2\theta}\right\},$$

or
$$x = r\sin qt + l\left\{1 - \sqrt{1 - \frac{r^2}{l^2}\cos^2 qt}\right\}.$$

Using the approximation that $\sqrt{1-\alpha} = 1 - \frac{1}{2}\alpha$ if α is small enough we have

$$x = r \sin qt + \frac{r^2}{2l} \cos^2 qt.$$

But we know that $2\cos^2 qt - 1 = \cos 2qt$. (See Art. 109.)

Hence $$\mathbf{x = r \sin qt + \frac{r^2}{4l} \cos (2qt) - \frac{r^2}{4l}} \quad \ldots\ldots\ldots(1).$$

We see that there is a fundamental simple harmonic motion, and its **octave** of much smaller amplitude.

Find $\frac{dx}{dt}$ and also $\frac{d^2x}{dt^2}$. This latter is

$$\text{acceleration} = -rq^2 \sin qt - \frac{r^2q^2}{l} \cos 2qt.$$

It will be seen that the relative importance of the octave term is four times as great in the acceleration as it was in the actual motion. We may, if we please, write θ again for $qt + \frac{\pi}{2}$ and get

$$\frac{d^2x}{dt^2} = rq^2 \cos \theta + \frac{r^2q^2}{l} \cos 2\theta.$$

When $\theta = 0$, the acceleration is $rq^2 + \frac{r^2q^2}{l}$.

When $\theta = 90°$, the acceleration is $-\frac{r^2q^2}{l}$.

When $\theta = 180°$, the acceleration is $-rq^2 + \frac{r^2q^2}{l}$,

(q is $2\pi f$, where f is the frequency or number of revolutions of the crank per second).

If three points be plotted showing displacement x and acceleration at these places, it is not difficult by drawing a curve through the three points to get a sufficiently accurate idea of the whole diagram. Perhaps, as to a point near the middle, it might be better to notice that when the angle OPQ is 90°, as P is moving uniformly and the rate of change of the angle Q is zero, there is no acceleration of Q just then. This position of Q is easily found by construction.

The most important things to recollect are (1) that accelerations, and therefore the forces necessary to cause motion, are four times as great if the frequency is doubled, and nine times as great if the frequency is trebled; (2) that the relative importance of an overtone in the motion is greatly exaggerated in the acceleration.

122. Take any particular form of **link motion or radial valve gear** and show that the motion of the valve is always very nearly (t being time from beginning of piston stroke or qt being angle passed through by crank from a dead point),

$$x = a_1 \sin(qt + \epsilon_1) + a_2 \sin(2qt + \epsilon_2) \ldots\ldots\ldots\ldots (1).$$

(There is a very simple method of obtaining the terms a_1 and a_2 by inspection of the gear.) When the overtone is neglected, a_1 is the half travel of the valve and ϵ_1 is the angle of advance. In a great number of radial valve gears we find that $\epsilon_2 = 90°$. The best way of studying the effect produced by the octave or overtone is to draw the curve for each term of (1) on paper by the method of Fig. 63, and then to add the ordinates together. If we subtract the outside lap L from x it is easy to see where the point of cut-off is, and how much earlier and quicker the cut-off is on account of this octave or kick in the motion of the valve.

In an example take $a_1 = 1$, $\epsilon_1 = 40°$, $a_2 = \cdot 2$, $\epsilon_2 = 90°$.

The practical engineer will notice that although the octave is good for one end of the cylinder it is not good for the other, so that it is not advisable to have it too great. We may utilize this fact in obtaining more admission in the up stroke of modern vertical engines; we may cause it to correct the inequality due to shortness of connecting rod.

Links and rods never give an important overtone of frequency 3 to 1. **It is always 2 to 1.**

In Sir F. Bramwell's gear the motion of the valve is, by the agency of spur wheels, caused to be

$$x = a_1 \sin(qt + \epsilon_1) + a_2 \sin(3qt + \epsilon_2) \ldots\ldots\ldots (2).$$

Draw a curve showing this motion when

$$a_1 = 1\cdot 15 \text{ inch}, \ \epsilon_1 = 47°, \ a_2 = \cdot 435, \ \epsilon_2 = 62°.$$

If the outside lap is 1 inch and there is no inside lap, find the positions of the main crank when cut-off, release, cushioning and admission occur. Show that this gear and any gear giving an overtone with an odd number of times the fundamental frequency, acts in the same way on both ends of the cylinder.

123. If $\quad x = a_1 \cos(q_1 t + \epsilon_1) + a_2 \cos(q_2 t + \epsilon_2) \ldots\ldots(3),$

where $\quad q_1 = 2\pi f_1$ and $q_2 = 2\pi f_2,$

there being two frequencies; this is not equivalent to one S. H. motion. Suppose a_1 to be the greater. The graphical method of study is best. We have two cranks of lengths a_1 and a_2 rotating with different angular velocities, so that the effect is as if we had a crank A rotating with the average angular velocity of a_1, but alternating between the lengths $a_1 + a_2$ and $a_1 - a_2$; always nearer a_1's position than a_2's; in fact, oscillating on the two sides of a_1's position. If q_1 is nearly the same as q_2 we have the interesting effect like **beats in music***.

Thus tones of pitches 100 and 101 produce 1 beat per second. The analogous beats are very visible on an incandescent lamp when two alternating dynamo-electrical machines are about to be coupled up together. Again, **tides of the sea,** except in long channels and bays, follow nearly the S. H. law; a_1 is produced by the moon and a_2 by the sun if $a_1 = 2{\cdot}1\, a_2$, so that the height of a spring tide is to the height of a neap tide as 3·1 to 1·1. The times of *full* are times of *lunar* full. The actual tide phase never differs more than 0·95 lunar hour from lunar tide; 0·95 lunar hour = 0·98 solar.

124. A Periodic Function of the time is one which becomes the same in every particular (its actual value, its rate of increase, &c.) after a time T. This T is called the

* Analytically. Take $\cos(2\pi f_2 t + \epsilon_2) = \cos\{2\pi f_1 t - 2\pi(f_1 - f_2)t + \epsilon_2\}$,

therefore $\quad x = r\cos(2\pi f_1 t + \theta),$

where $\quad r^2 = a_1{}^2 + a_2{}^2 + 2a_1 a_2 \cos\{2\pi(f_1 - f_2)\,t + \epsilon_1 - \epsilon_2\},$

and the value of $\tan\theta$ is easily written out.

periodic time and its reciprocal is called f the frequency. Algebraically the definition of a periodic function is

$$f(t) = f(t + nT),$$

where n is any positive or negative integer.

125. Fourier's Theorem can be proved to be true. It states that any periodic function whose complete period is T (and q is $2\pi/T$ or $2\pi f$) is really equivalent to the sum of a constant term and certain sine functions of the time

$$f(t) = A_0 + A_1 \sin(qt + E_1) + A_2 \sin(2qt + E_2) + A_3 \sin(3qt + E_3) + \&c. \ldots\ldots\ldots (1).$$

In the same way, the note of any organ pipe or fiddle string or other musical instrument consists of a fundamental tone and its overtones. (1) is really the same as

$$f(t) = A_0 + a_1 \sin qt + b_1 \cos qt + a_2 \sin 2qt + b_2 \cos 2qt + \&c.,$$

if $a_1^2 + b_1^2 = A_1^2$ and $\tan E_1 = \dfrac{b_1}{a_1}$, &c.†

126. A varying magnetic field in the direction x follows the law $X = a \sin qt$ where t is time. Another in the direction y, which is at right angles to x, follows the law

$$Y = a \cos qt.$$

At any instant the resultant field is

$$R = \sqrt{X^2 + Y^2} = a = \text{a constant}$$

making with y the angle θ, where $\tan \theta = Y/X$, or $\theta = qt$.

Hence the effect produced is that we have a constant field R rotating with angular velocity q.

When the fields are

$$X = a_1 \sin(qt + \epsilon_1) \text{ and } Y = a_2 \sin(qt + \epsilon_2),$$

it is better to follow a graphical method of study. The resultant field is represented in amount and direction by the radius vector of an ellipse, describing equal areas in equal times.

Let OX and OY, fig. 69, be the two directions mentioned. Let OA_1 in the direction $OX = a_1$. With OA_1 as radius describe

a circle. Let $YO0$ be the angle e_1. Divide the circle into many equal parts starting at 0 and naming the points of division 0, 1, 2, 3, &c. Draw lines from these points parallel to OY. Let OA_2 in the direction OY be a_2. Describe a circle with OA_2 as radius. Set off the angle $X'O0'$ as e_2 and

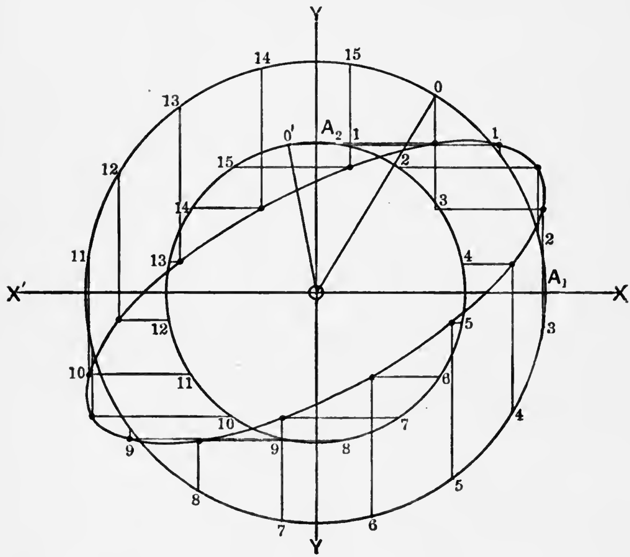

Fig. 69.

divide this circle at 0′, 1, 2, 3, 4, &c., into the same number of parts as before. Let lines be drawn from these points parallel to OX, and where each meets the corresponding line from the other circle we have a point whose radius vector at any instant represents, in direction and magnitude, the resultant magnetic field.

If OX and OY are not at right angles to one another, the above instructions have still to be followed.

If we divide the circle OA_2 into only half the number of parts of OA_1 we have the combination of $X = a_1 \sin(qt + \epsilon_1)$ and $Y = a_2 \sin(2qt + \epsilon_2)$.

If we wish to see the combination $X =$ any periodic function and Y any other periodic function, let the curve

from M_2 to N_2 show Y, M_2N_2 being the whole periodic time; and let the curve from M_1 to N_1 show X, the vertical distance M_1N_1

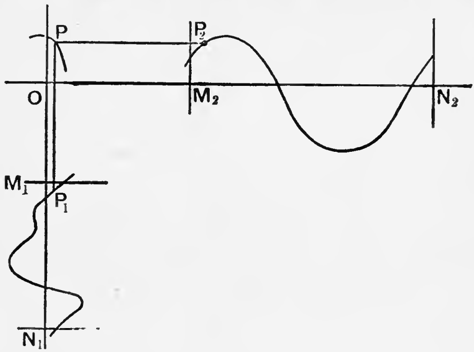

Fig. 70.

being the whole periodic time. If P_1 and P_2 are points on the two curves at identical times, let the horizontal line from P_2 meet the vertical line from P_1 in P. Then at that instant **OP represents the resultant field in direction and magnitude.**

Carry out this construction carefully. It has a bearing on all sorts of problems besides problems on rotating magnetic fields.

127. The **area of a sine curve** for a whole period or for any number of whole periods is zero. This will be evident if one draws the curve. By actual calculation; let s be an integer and $q = \frac{2\pi}{T}$,

$$\int_0^T \sin sqt \,.\, dt = -\frac{1}{sq}\Big[\cos sqt\Big]_0^T = -\frac{1}{sq}\left(\cos s\frac{2\pi}{T}T - \cos 0\right) = 0,$$

because $\cos s\frac{2\pi}{T}T$ or $\cos s2\pi = 1$ and $\cos 0 = 1$.

$$\text{Again, } \int_0^T \cos sqt \,.\, dt = \frac{1}{sq}\Big[\sin sqt\Big]_0^T$$
$$= \frac{1}{sq}\left(\sin s\frac{2\pi}{T}T - \sin 0\right) = 0,$$

because $\sin s\frac{2\pi}{T}T = \sin s2\pi = 0$ and $\sin 0 = 0$.

128. If the ordinates of two sine curves be multiplied together to obtain the ordinate of a new curve: the area of it is 0 for any period which is a multiple of each of their periods. Thus if s and r are any integers

$$\int_0^T \sin sqt \, . \cos rqt \, . \, dt = 0 \quad \text{............(1)},$$

$$\int_0^T \sin sqt \, . \sin rqt \, . \, dt = 0 \quad \text{............(2)},$$

$$\int_0^T \cos sqt \, . \cos rqt \, . \, dt = 0 \quad \text{............(3)}.$$

These ought to be tried carefully. 1st as Exercises in Integration. 2nd Graphically. The student cannot spend too much time on looking at these propositions from many points of view. He ought to see very clearly why the answers are 0. The functions in (1) and (2) and (3) really split up into single sine functions and the integral of each such function is 0. Thus

$$2 \sin sqt \, . \cos rqt = \sin (s + r) \, qt + \sin (s - r) \, qt,$$

and by Art. 127, each of these has an area 0.

The physical importance of the proposition is enormous. Now if $s = r$ the statements (2) and (3) are untrue, but (1) continues true. For

$$\int_0^T \sin^2 sqt \, . \, dt = \int_0^T \cos^2 sqt \, . \, dt = \tfrac{1}{2} T \quad \text{......(4)},$$

whereas (1) becomes the integral of $\frac{1}{2} \sin 2sqt$ which is 0. (4) ought to be worked at graphically as well as by mere integration. Recollecting the trigonometrical fact that

$$\cos 2\theta = 2 \cos^2 \theta - 1 \ \text{ or } \ 1 - 2 \sin^2 \theta,$$

and therefore that

$$\cos^2 qt = \tfrac{1}{2} \cos 2qt + \tfrac{1}{2}, \ \sin^2 qt = \tfrac{1}{2} - \tfrac{1}{2} \cos 2qt,$$

the integration is easy and the student ought to use this method as well as the graphical method.

129. To illustrate the work graphically. Let OC, fig. 71, be T. Taking $s = 2$, the curve $OPQRSC$ represents $\sin sqt$.

Its maximum and minimum heights are 1. Now note that $\sin^2 sqt$ is always + and it is shown in $OP'Q_1R_1S_1C$. It fluctuates between 0 and 1 and its average height is $\frac{1}{2}$ or

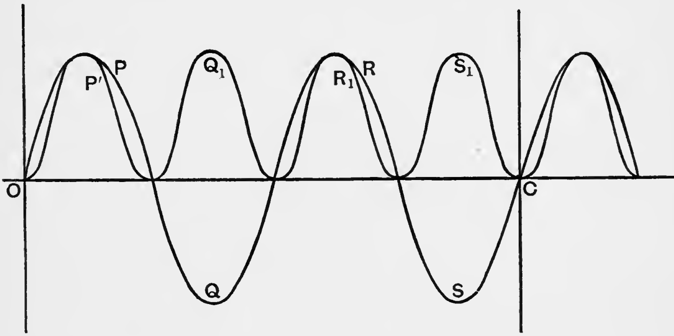

Fig. 71.

the area of the whole curve from O to C is $\frac{1}{2}T$. The fact that **the average value of sin sqt. × sin rqt is 0, but that the average value of sin sqt × sin sqt is $\frac{1}{2}$, is one of the most important in practical engineering work.**

130. Illustration in Electricity. An electric dynamometer has two coils; one fixed, through which, let us suppose, a current C flows; the other moveable, with a current c. At any instant the resultant force or couple is proportional to Cc and enables us to measure Cc. But if C and c vary rapidly we get the *average* value of Cc. Prof. Ayrton and the author have carried out the following beautifully illustrative experiment. They sent a current through the fixed coil which was approximately, $C = C_0 \sin 2\pi ft$. This was supplied by an alternating dynamo machine. Through the other coil they sent a current, $c = c_0 \sin 2\pi f't$ whose frequency could be increased or diminished. It was very interesting to note (to the average practical engineer it was uncanny, unbelievable almost) that although great currents were passing through the two coils, there was no average force—in fact there was no *reading* as one calls it in the laboratory. Suppose f was 100 per second, f' was gradually increased from say 10 to 20, to 30 to 40 to 49. Possibly about 49 to

51 a vague and uncertain sort of action of one coil on the other became visible, a thing not to be measured, but as f' increased the action ceased. No action whatever as f' became 60, 70, 80, 90, 97, 98, 99, but as f' approached 100 there was no doubt whatever of the large average force; a reading could be taken and it represented according to the usual scale of the instrument $\frac{1}{2}C_0c_0$; when f' increased beyond 100 the force suddenly ceased and remained steadily 0 until f' became 200 when there was a small force to be measured; again it ceased suddenly until f' became 300, and so on. We know that if C and c had been true sine functions there would have been absolutely no force except when the frequencies were exactly equal. In truth, however, the octaves and higher harmonics were present and so there were slight actions when f and f' were as 2 to 1 or 1 : 2 or 1 : 3, &c. This is an extremely important illustration for all electrical engineers who have to deal with alternating currents of electricity.

131. Exercise in Integration. C and c being alternating currents of electricity. When $C = C_0 \sin qt$ and $c = c_0 \sin(qt \pm e)$ and these two currents flow through the two coils of an electro-dynamometer, the instrument records $\frac{1}{2}C_0c_0 \,.\, \cos e$ as this is the average value of the product Cc.

When C and c are the same, that is, when the same current $C = C_0 \sin(qt + e)$ passes through both coils, the instrument records the average value of $C^2 \,.\, dt$, or

$$\frac{1}{T}\int_0^T C_0^2 \sin^2(qt+e)\,.\,dt \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

which we know to be $\frac{1}{2}C_0^2$. The square root of any such reading is usually called the **effective current,** so that $\frac{1}{\sqrt{2}}C_0$ is what is known as the effective value of $C_0 \sin qt$. **Effective current is defined as the square root of mean square of the current.** Thus when an electrical engineer speaks of an alternating current of 100 amperes he means that the effective current is 100 amperes or that $C = 141{\cdot}4 \sin(qt + \alpha)$. Or the voltage 1000 means

$$v = 1414 \sin(qt + \beta).$$

Exercise. What is the *effective* value of

$$a_0 + A_1 \sin(qt + \epsilon_1) + A_2 \sin(2qt + \epsilon_2) + \&c.?$$

Notice that only the squares of terms have an average value, the integral of any other product being 0 during a complete period. Answer: $\sqrt{a_0^2 + \frac{1}{2}(A_1^2 + A_2^2 + \&c.)}$.

Observe the small importance of small overtones.

If $v = \dfrac{4v_0}{\pi}(\sin qt + \frac{1}{3}\sin 3qt + \frac{1}{5}\sin 5qt\ \&c.)$ ....(2),

we shall see from Art. 135 that this is the Fourier expression for what is shown in the curve (fig. 72) the distance OM being called v_0 and the distance OQ being the periodic time T, where $q = \dfrac{2\pi}{T}$, and v is measured upwards from the line OQ.

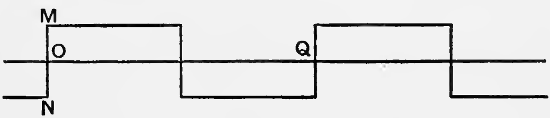

Fig. 72.

The effective $v = \dfrac{4v_0}{\pi\sqrt{2}}\sqrt{1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \&c.}$ ....(3).

Again in fig. 73, where $PM = v_0$ and $OQ = T$,

$$v = \frac{8v_0}{\pi^2}(\sin qt - \tfrac{1}{9}\sin 3qt + \tfrac{1}{25}\sin 5qt - \&c.) \ldots(4).$$

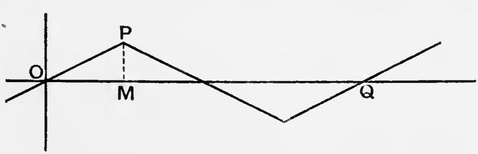

Fig. 73.

The effective $v = \dfrac{8v_0}{\pi^2\sqrt{2}}\sqrt{1 + \frac{1}{81} + \frac{1}{625} + \&c.}$ ....(5).

Again note the small importance of everything except the fundamental term.

Exercise. If $C = C_0 + A_1 \sin qt + B_1 \cos qt$
$$+ A_2 \sin 2qt + B_2 \cos 2qt + \&c. \ldots (6),$$
and if
$$c = c_0 + a_1 \sin qt + b_1 \cos qt + a_2 \sin 2qt + b_2 \cos 2qt + \&c. \ldots (7).$$
$$\text{Average } Cc = C_0 c_0 + \tfrac{1}{2}(A_1 a_1 + B_1 b_1 + A_2 a_2 + B_2 b_2 + \&c.) \ldots (8).$$

It will be seen that there are no terms like $A_2 b_2$ or $A_2 b_3$.

132. Let AB and BC be parts of an electric circuit. In AB let the resistance be R and let there be no self-induction. In BC let the resistance be r and let there be self-induction l. If $C = C_0 \sin qt$ is the current passing. Let V_{AB} &c. represent the voltage between the points A and B, &c. Let $\bar{V}_{AB}$ mean the *effective* voltage between A and B.

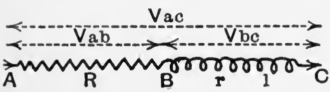

Fig. 74.

$$V_{AB} = RC_0 \sin qt,$$
$$V_{BC} = C_0 \sqrt{r^2 + l^2 q^2} \sin \left(qt + \tan^{-1} \frac{lq}{r}\right), \text{ see Art. 118},$$
$$V_{AC} = C_0 \sqrt{(R+r)^2 + l^2 q^2} \sin \left(qt + \tan^{-1} \frac{lq}{R+r}\right),$$
$$\bar{V}_{AB} = \frac{1}{\sqrt{2}} RC_0, \quad \bar{V}_{BC} = \frac{1}{\sqrt{2}} rC_0 \sqrt{1 + \frac{l^2 q^2}{r^2}},$$
$$\bar{V}_{AC} = \frac{1}{\sqrt{2}} (R+r) C_0 \sqrt{1 + \frac{l^2 q^2}{(R+r)^2}}.$$

Observe that $\bar{\mathbf{V}}_{\mathbf{AC}}$ **is always less than** $\bar{\mathbf{V}}_{\mathbf{AB}} + \bar{\mathbf{V}}_{\mathbf{BC}}$, **or the effective voltage between A and C is always less than the sum of the effective voltages between A and B and between B and C.**

Thus take $C_0 = 141{\cdot}4$, $R = 1$, $r = 1$, $lq = 1$, and illustrate a fact that sometimes puzzles electrical engineers.

133. Rule for developing any arbitrary function in a Fourier Series.

†The function may be represented as in fig. 75, PE represents the value of y at the time t which is represented by

OE, OC represents the whole periodic time T. At C the curve is about to repeat itself. (Instead of using the letter

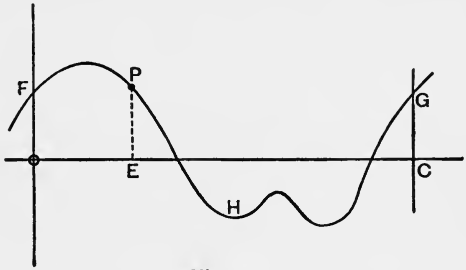

Fig. 75.

t we may use x or any other. We have functions which are periodic with respect to space for example.) Assume that y can be developed as

$$y = a_0 + a_1 \sin qt + b_1 \cos qt + a_2 \sin 2qt + b_2 \cos 2qt + a_3 \sin 3qt + b_3 \cos 3qt + \&c. \quad \ldots(1),$$

where $$q = \frac{2\pi}{T}.$$

It is evident from the results given in Art. 127 that $\mathbf{a_0}$ **is the average height of the curve,** or the average value of y. This can be found as one finds the average height of an indicator diagram. Carry a planimeter point from O to $FPHGCO$, and divide the whole area thus found by OC. If we have not drawn the curve; if we have been given say 36 equidistant values of y, add up and divide by 36. The reason is this; the area of the whole curve, or the integral of y between the limits 0 and T, is a_0T, because the integral of any other term such as $a_1 \sin qt$ or $b_3 \cos 3qt$ is 0. In fact

$$\int_0^T \sin sqt \,.\, dt \text{ or } \int_0^T \cos sqt \,.\, dt \text{ is } 0,$$

if s is an integer.

a_1 is twice the average height of the curve which results from multiplying the ordinates y by the corresponding

ordinate of $\sin qt$; for, multiply (1) all across by $\sin qt$, and integrate from 0 to T, and we have by Art. 128

$$\int_0^T y \,.\, \sin qt \,.\, dt = 0 + a_1 \int_0^T \sin^2 qt \,.\, dt + 0 + 0 + \&c. = \tfrac{1}{2} a_1 T \ldots (1);$$

dividing by T gives the average value, and twice this is evidently a_1. Similarly

$$\int_0^T y \,.\, \cos qt \,.\, dt = \tfrac{1}{2} b_1 T \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

In fact, by the principles of Art. 128, **a_s and b_s are twice the average values of y sin sqt and y . cos sqt,** or

$$\left.\begin{aligned} a_s &= \frac{2}{T}\int_0^T y \,.\, \sin sqt \,.\, dt \\ b_s &= \frac{2}{T}\int_0^T y \,.\, \cos sqt \,.\, dt \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (3).$$

134. In the *Electrician* newspaper of Feb. 5th, 1892, the author gave clear instructions for carrying out this process numerically when 36 numbers are given as equidistant values of y.

In the same paper of June 28th, 1895, the author described a **graphical method** of finding the coefficients. The graphical method is particularly recommended for developing any arbitrary function.

Students who refer to the original paper will notice that the abscissae are very quickly obtained and the curves drawn.

In this particular case we consider the original curve showing y and time, to be wrapped round a circular cylinder whose circumference is the periodic time. The curve is projected upon a diametral plane passing through $t = 0$. Twice the area of the projection divided by the circumference of the cylinder is a_1. Projected upon a plane at right angles to the first, we get b_1 in the same way. When the curve is wrapped round s times instead of once, and projected on the two diametral planes, twice the areas of each of the

two projections divided by s times the circumference of the cylinder give a_s and b_s.*

Prof. Henrici's Analyzers, described in the *Proceedings of the Physical Society*, give the coefficients rapidly and accurately. The method of Mr Wedmore, published in the *Journal of the Institution of Electrical Engineers*, March 1896, seems to me very rapid when a column of numbers is given as equidistant values of y.

135. When a periodic function is graphically represented by straight lines like fig. 72 or fig. 73 we may obtain the development by direct integration. Thus in fig. 76, the Electrician's Make and Break Curve;

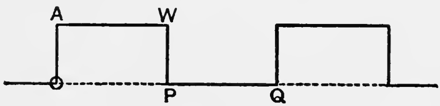

Fig. 76.

$$y = OA, \text{ or } 2v_0 \text{ say, from } t = 0 \text{ to } t = OP = \tfrac{1}{2}T;$$

$$y = 0 \text{ from } t = \tfrac{1}{2}T \text{ or } OP, \text{ to } t = T \text{ or } OQ.$$

Evidently $$a_0 = v_0,\ q = \frac{2\pi}{T};$$

$$a_s = \frac{2}{T}\int_0^{\frac{1}{2}T} 2v_0 \,.\, \sin sqt \,.\, dt,\quad b_s = \frac{2}{T}\int_0^{\frac{1}{2}T} 2v_0 \cos sqt \,.\, dt,$$

$$a_s = -\frac{4v_0}{T} \cdot \frac{T}{2s\pi}\left[\cos s \,.\, \frac{2\pi}{T}t\right]_0^{\frac{1}{2}T},\quad b_s = \frac{4v_0}{T} \cdot \frac{T}{2s\pi}\left[\sin s \,.\, \frac{2\pi}{T}t\right]_0^{\frac{1}{2}T},$$

* The method is based upon this, that

$$a_s = \frac{2}{T}\int_0^T y \,.\, \sin sqt \,.\, dt = -\frac{2}{sqT}\int y \,.\, d\,(\cos sqt) = -\frac{1}{s\pi}\int y \,.\, d\,(\cos sqt).$$

Drawing a complete curve of which y (at the time t) is the ordinate and $\cos sqt$ is the abscissa, we see that its area as taken by a Planimeter divided by $s\pi$ gives a_s. This graphical method of working is made use of in developing arbitrary functions in series of other normal forms than sines and cosines, such as Spherical Zonal Harmonics and Bessels.

By the above method, $b_s = \dfrac{1}{s\pi}\displaystyle\int y \,.\, d\,(\sin sqt)$.

$$a_s = -\frac{2v_0}{s\pi}(\cos s\pi - \cos 0) = -\frac{2v_0}{s\pi}\begin{pmatrix} 0 \text{ if } s \text{ is even} \\ -2 \text{ if } s \text{ is odd}\end{pmatrix}$$

$$= \frac{4v_0}{s\pi} \text{ if } s \text{ is odd,}$$

$$b_s = \frac{2v_0}{s\pi}(\sin s\pi - \sin 0) = 0.$$

Hence the function shown in fig. 76 becomes

$$y = v_0 + \frac{4v_0}{\pi}(\sin qt + \tfrac{1}{3}\sin 3qt + \tfrac{1}{5}\sin 5qt + \&c.) \ldots\ldots(1).$$

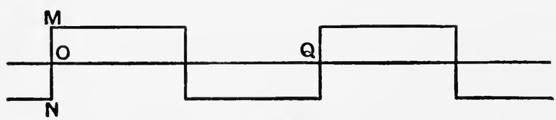

Fig. 77.

If the origin is half-way between O and A (fig. 76), as in fig. 77, so that instead of what the electricians call a make and break we have v_0 constant for half a period, then $-v_0$ for the next half period, that is, reversals of y every half period, we merely subtract v_0, then

$$y = \frac{4v_0}{\pi}(\sin qt + \tfrac{1}{3}\sin 3qt + \tfrac{1}{5}\sin 5qt + \&c.) \ldots(2).$$

Let the origin be half-way between O and P, fig. 76; the t of (1) being put equal to a new $t + \frac{1}{4}T$,

$\sin sqt$ where s is odd, becomes $\sin sq(t + \frac{1}{4}T)$,

or $$\sin s\frac{2\pi}{T}(t + \tfrac{1}{4}T) \text{ or } \sin\left(sqt + s\frac{\pi}{2}\right),$$

where $s = 1, 5, 9, 13$ &c. this becomes $\cos sqt$,

„ $s = 3, 7, 11, 15$ &c. „ „ $-\cos sqt$,

and consequently with the origin at a point half-way between O and P,

$$y = v_0 + \frac{4v_0}{\pi}(\cos qt - \tfrac{1}{3}\cos 3qt + \tfrac{1}{5}\cos 5qt - \tfrac{1}{7}\cos 7qt + \&c.).$$

136. To represent a periodic function of x for all values of x it is necessary to have series of terms **each of which is itself a periodic function.** The Fourier series is the simplest of these.

137. If the values of y, a function of x, be given for all values of x between $x=0$ and $x=c$; y can be expanded in **a series of sines only** or **a series of cosines only.** Here we regard the given part as only half of a complete periodic function and we are not concerned with what the series represents when x is less than 0 or greater than c. In the previous case y was completely represented for all values of the variable.

I. Assume $y=a_1 \sin qx + a_2 \sin 2qx + \&c.$ where $q=\pi/c$.

Multiply by $\sin sqx$ and integrate between the limits 0 and c. It will be found that all the terms disappear except $\int_0^c a_s \sin^2 sqx \, . \, dx$ which is $\frac{1}{2}a_s c$, so that a_s is twice the average value of $y \, . \, \sin sqx$.

Thus let y be a constant m, then

$$a_s = \frac{2}{c}\int_0^c m \sin sqx \, . \, dx = -\frac{2m}{csq}\Big[_0^c \cos sqx\Big]$$

$$= -\frac{2m}{csq}(\cos s\pi - 1) = \frac{4m}{s\pi} \text{ if } s \text{ is odd,}$$

$$= 0 \text{ if } s \text{ is even.}$$

Hence $$m = \frac{4m}{\pi}(\sin qx + \tfrac{1}{3}\sin 3qx + \tfrac{1}{5}\sin 5qx + \&c.)^*.$$

II. Assume $y=b_0+b_1\cos qx + b_2 \cos 2qx + \&c.$ Here b_0 is evidently the mean value of y from $x=0$ to $x=c$. In the

---

* *Exercise.* Develope $y=mx$ from $x=0$ to $x=c$ in a series of sines.

$$mx = a_1 \sin qx + a_2 \sin 2qx + \&c., \text{ where } q=\frac{\pi}{c},$$

a_s is $$\frac{2}{c}\int_0^c mx \, . \sin sqx \, . \, dx = \frac{2m}{s^2q^2c}\Big[_0^c \sin sqx - sqx \, . \cos sqx\Big].$$

For this integral refer to (70) page 365.

Hence $$mx = \frac{2mc}{\pi}\left(\sin\frac{\pi}{c}x - \frac{1}{2}\sin\frac{2\pi}{c}x + \frac{1}{3}\sin\frac{3\pi}{c}x - \&c.\right).$$

same way as before we can prove that b_s is twice the average value of $y \cos sqx$*.

138. In Art. 118 we gave the equation for an electric circuit. The evanescent term comes in as before but we shall neglect it. Observe that if V is not a simple sine function of t, but a complicated periodic function, each term of it gives rise to a term in the current, of the same period. Thus if

$$V = V_0 + \Sigma\, V_s \sin(sqt + e_s)^{\dagger} \quad\ldots\ldots\ldots\ldots(1),$$

$$C = \frac{V_0}{R} + \Sigma \frac{V_s}{\sqrt{R^2 + L^2 s^2 q^2}} \sin\left(sqt + e_s - \tan^{-1}\frac{sLq}{R}\right) \ldots(2).$$

If Lq is very large compared with R we may take

$$C = \frac{V_0}{R} - \Sigma \frac{V_s}{Lsq} \cos(sqt + e_s) \quad\ldots\ldots\ldots\ldots(3).$$

Thus, taking the make and break curve for V, fig. 76,

$$V = V_0 + \frac{4V_0}{\pi}(\sin qt + \tfrac{1}{3}\sin 3qt + \&c.) \quad\ldots\ldots\ldots\ldots(4),$$

$$C = \frac{V_0}{R} - \frac{2V_0 T}{\pi^2 L}(\cos qt + \tfrac{1}{9}\cos 3qt + \tfrac{1}{25}\cos 5qt + \&c.) \quad\ldots(5),$$

which is shown by the curve of fig. 73, 0 being at $\dfrac{T}{4}$.

139. When **electric power** is supplied to a house or contrivance, the power in watts is the average value of CV where C is current in amperes and V the voltage.

---

* Thus let $y = mx$ between $x = 0$ and $x = c$. Evidently $b_0 = \frac{1}{2}mc$, and we find $$y = \frac{m\pi}{2} - \frac{4m}{\pi}\left(\cos qx + \frac{1}{9}\cos 3qx + \frac{1}{25}\cos 5qx + \&c.\right).$$

There are many other normal forms in which an arbitrary function of x may be developed. Again, even of sines or cosines there are other forms than those given above. For example, if we wish generally to develope y a function of x between 0 and c as $y = \Sigma a_m \sin \alpha_m x$ by the Fourier method, the essential principle of which is $\int_0^c \sin \alpha_n x \,.\, \sin \alpha_m x \,.\, dx = 0$, where m and n are different; we must have α_m and α_n, roots of $\dfrac{\alpha c \cos \alpha c}{\sin \alpha c} = s$. In the ordinary Fourier series s is ∞.

Let $V = V_0 \sin qt$ and $C = C_0 \sin (qt - e)$.

Then $P = \frac{1}{2} C_0 V_0 \cos e$†, or half the product of the amplitudes multiplied by the cosine of the lag. When the power is measured by passing C through one coil of a dynamometer and allowing V to send a current c through the other coil, if this coil's resistance is r and self-induction l

$$c = \frac{V_0}{\sqrt{r^2 + l^2 q^2}} \sin \left(qt - \tan^{-1} \frac{lq}{r}\right) \quad \text{.........(6).}$$

What is really measured therefore is the average value of Cc, or

$$\frac{1}{2} \frac{C_0 V_0}{\sqrt{r^2 + l^2 q^2}} \cos \left(e - \tan^{-1} \frac{lq}{r}\right).$$

Usually in these special instruments, large non-inductive resistances are included in the fine wire circuit and we may take it that lq is so small in comparison with r that its square may be neglected. If so, then

$$\frac{\text{apparent power}}{\text{true power}} = \frac{\cos \left(e - \tan^{-1} \frac{lq}{r}\right)}{\cos e}.$$

Observe that $\tan^{-1} \frac{lq}{r}$ is a very small angle, call it α,

$$\frac{\text{apparent power}}{\text{true power}} = \frac{\cos e \cos \alpha + \sin e \sin \alpha}{\cos e} = \cos \alpha + \sin \alpha \,.\, \tan e.$$

Now $\cos \alpha$ is practically 1, and $\sin \alpha$ is small, and at first sight it might seem that we might take the answer as nearly 1.

But if e is nearly 90° its tangent may be exceedingly large and **the apparent power may be much greater than the true power.**

It is seldom however that e approaches 90° unless in coils of great diameter with no iron present, and precautions taken to avoid eddy currents. Even when giving power to a choking coil or unloaded transformer, the effect of hysteresis is to cause e not to exceed 74°.

140. True Power Meter. Let EG and GD be coils wound together as the fixed part of a dynamometer, and let

DB be the moveable coil. The current $C+c$ passes from E to G. Part of it c goes along the non-inductive resistance GF which has a resistance R. The part C flows from G to D and D to B and through the house or contrivance. The instantaneous value of $Rc.C$ is the instantaneous power.

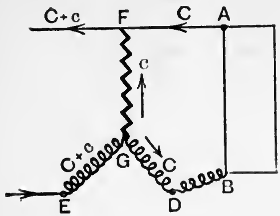

Fig. 78.

The coils EG and GD are carefully adjusted so that when $c=0$ and the currents are continuous currents, there shall be no deflection of the moveable coil DB. Hence the combined action of $C+c$ in EG and of C in GD upon C in DB is force or torque proportional to cC, and hence the reading of the instrument is proportional to the power. With varying currents also there will be no deflection if there is no metal near capable of forming induced currents.

141. The student ought to get accustomed to translating into ordinary language such a statement as (1) of Art. 119. Having done so, consider a mass of W lb.* hanging from a spring whose stiffness is such that a force of 1 lb. elongates it h feet. If there is vibration; when W is at the level CC, fig. 79, x feet below (we imagine it moving downwards) its position of equilibrium OO, the force urging it to the position of equilibrium is $x \div h$ pounds, and as the moving mass is $\frac{W}{g}$ (neglect the mass of the spring itself or consider one-third of it as being added to the moving body),

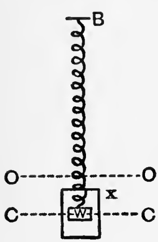

Fig. 79.

$$\frac{W}{g} \times \text{the acceleration} = \frac{x}{h}.$$

The acceleration $= \frac{xg}{Wh}$. The acceleration is then pro-

* The name W lb. is the weight of a certain quantity of stuff; the inertia of it in Engineers' units is $W \div 32{\cdot}2$.

portional to x, and our $\frac{g}{Wh}$ stands for q^2 in (1) of Art. 119, and (2) shows the law connecting x and t.

Notice carefully that the + sign in (1) is correct. The body is moving downwards and x is increasing, so that dx/dt is positive. But $\frac{d^2x}{dt^2}$ is negative, the body getting slower in its motion as x increases.

142. Imagine the body to be retarded by a force which is proportional to its velocity, or $b\frac{dx}{dt}$. Observe that this acts as $\frac{x}{h}$ acts, that is upwards, towards the position of equilibrium.

Hence we may write

$$\frac{W}{g}\frac{d^2x}{dt^2} + b\frac{dx}{dt} + \frac{x}{h} = 0 \quad\text{.................}(1).$$

We shall presently see what law now connects x and t in this **damped vibration.**

143. Suppose that in the last exercise, when the body is displaced x feet downwards, its point of support B is also y feet below its old position. The spring is really only elongated by the amount $x-y$, and the restoring force is $\frac{x-y}{h}$. Consequently (1) ought to be

$$\frac{W}{g}\frac{d^2x}{dt^2} + b\frac{dx}{dt} + \frac{x}{h} = \frac{y}{h} \quad\text{.................}(2).$$

Now imagine that the motion y is given as a function of the time, and we are asked to find x as a function of the time. y gives rise to what we call a **forced vibration.** If $y=0$ we have the natural vibrations only.

We give this, not for the purpose of solving it just now, although it is not difficult, but for the purpose of familiarizing the student with differential equations and inducing him to translate them into ordinary language.

144. Notice that if the **angular** distance of a rigid body from its position of equilibrium is θ, if I is its moment of inertia about an axis through the centre of gravity, if $H\theta$ is the sum of the moments of the forces of control about the same axis, and if $F\frac{d\theta}{dt}$ is the moment of frictional forces which are proportional to velocity,

$$I\frac{d^2\theta}{dt^2}+F\frac{d\theta}{dt}+H\theta=H\theta, \quad \ldots\ldots\ldots\ldots(3),$$

if θ' is the forced angular displacement of the case to which the springs or other controlling devices are attached.

145. The following is a specially good example. Referring back to Example 1 of Art. 98, we had CR, the voltage in the circuit, connecting the coatings of the condenser. If we take into account self-induction L in this circuit, then the voltage v is

$$RC+L\frac{dC}{dt}=v \quad \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

We may even go further and say that if there is an alternator in the circuit, whose electromotive force is e at any instant (e, if a constant electromotive force would oppose C as shown in the figure)

$$RC+L\frac{dC}{dt}=v-e \quad \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

But we saw that the current $C=-K\frac{dv}{dt}$ .... (6).

Using this value of C in (5) we get

$$-RK\frac{dv}{dt}-LK\frac{d^2v}{dt^2}=v-e,$$

or

$$LK\frac{d^2v}{dt^2}+RK\frac{dv}{dt}+v=e \quad \ldots\ldots\ldots\ldots(7).$$

Now imagine that e is given as a function of the time and we are asked to find v as a function of the time.

e gives rise to what we call a **forced vibratory current** in the system. If $e=0$ we have the *natural* vibrations only of the system. Having v, (6) gives us C.

146. If (7) is compared with (2) or (3) we see at once **the analogy between a vibrating mechanical system and an electrical one.**

They may be put

$$\frac{W}{g}\frac{d^2x}{dt^2}+b\frac{dx}{dt}+\frac{x}{h}=\frac{y}{h}, \text{ Mechanical } \ldots\ldots\ldots(8),$$

$$L\frac{d^2v}{dt^2}+R\frac{dv}{dt}+\frac{v}{K}=\frac{e}{K}, \text{ Electrical } \ldots\ldots\ldots(9).$$

The mass $\frac{W}{g}$ corresponds with self-induction L.

The friction per foot per second b, corresponds with the resistance R.

The displacement x, corresponds with voltage v, or to be seemingly more accurate, v is Q the electric displacement divided by K.

The want of stiffness of the spring h corresponds with capacity of condenser K.

The forced displacement y corresponds with the forced E.M.F. of an alternator.†

147. The complete solution of (8) or (9), that is, the expression of x or v as a function of t, will be found to include:—

(1) The solution if y or e were 0.

This is **the natural vibration** of the system, which dies away at a rate which depends upon the mechanical friction in the one case and the electrical friction or resistance in the other case. We shall take up, later, the study of this vibration. It ought to be evident without explanation, that if y or e is 0, we have a statement of what occurs when the system is left to itself.

(2) The solution which gives the **forced vibrations** only.

The sum of these two is evidently the complete answer.†

148. Forced Vibration. As the Mechanical and Electrical cases are analogous, let us study that one about

which it is most easy to make a mental picture, the mechanical case. We shall in the first place assume no friction and neglect the natural vibrations, which are however only negligible when there is some friction. Then (8) becomes

$$\frac{d^2x}{dt^2} + \frac{g}{Wh}x = \frac{g}{Wh}y \text{ ....(10).}$$

Let $y = a \sin qt$ be a motion given to the point of support of the spiral spring which carries W; y may be any complicated periodic function, we consider one term of it.

We know that if y were 0, the natural vibration would be $x = b \sin\left(t\sqrt{\frac{g}{Wh}} + m\right)$, where b and m might have any values whatsoever. It is simpler to use n^2 for g/Wh as we have to extract its square root. n is 2π times the frequency of the natural vibrations of W. We had better write the equation as

$$\frac{d^2x}{dt^2} + n^2x = n^2y = n^2a \sin qt \text{ ....(11).}$$

Now try if there is a solution, $x = A \sin qt + B \cos qt$. If so, since $\frac{d^2x}{dt^2} = -Aq^2 \sin qt - Bq^2 \cos qt$; equating the coefficients of $\sin qt$ and also those of $\cos qt$, $-Aq^2 + n^2A = n^2a$, so that $A = \frac{n^2a}{n^2 - q^2}$, and $-Bq^2 + n^2B = 0$, so that $B = 0$ unless $n = q$. We see that we have the solution

$$x = \frac{n^2a}{n^2 - q^2} \sin qt \text{ ....(12).}$$

This shows that there is **a forced vibration** of W which is synchronous with the motion of the point of support; its amplitude being $\dfrac{1}{1 - \frac{q^2}{n^2}}$ times that of the point of support.

Now take a few numbers to illustrate this answer. Let $a = 1$, let $\frac{q}{n}$ be great or small. Thus $\frac{q}{n} = \frac{1}{10}$ means that the forced frequency is one tenth of the natural frequency.

$\frac{q}{n}$	Amplitude of W's motion	$\frac{q}{n}$	Amplitude of W's motion
·1	1·01	1	∞
·5	1·333	1·01	− 50
·8	2·778	1·03	− 16·4
·9	5·263	1·1	− 4·762
·95	10·26	1·5	− 0·8
·97	16·92	2·0	− 0·333
·98	25·25	5·0	− 0·042
·99	50·25	10·0	− 0·010

Note that when the forced frequency is a small fraction of the natural frequency, the forced vibration of W is a **faithful copy** of the motion of the point of support B; the spring and W move like a rigid body. When the forced is increased in frequency the motion of W is a **faithful magnification** of B's motion. As the forced gets nearly equal to the natural, the motion of W is an **enormous magnification** of B's motion. There is always some friction and hence the amplitude of the vibration cannot become infinite. When the forced frequency is greater than the natural, W is always **a half-period behind** B, being at the top of its path when B is at the bottom. When the forced is many times the natural, the motion of W gets to be **very small**; it is nearly at rest.

Men who design **Earthquake** recorders try to find a steady point which does not move when everything else is moving. For up and down motion, observe that in the last case just mentioned, W is like a steady point.

When the forced and natural frequencies are nearly equal, we have the state of things which gives rise to **resonance** in acoustic instruments; which causes us to fear for suspension bridges or rolling ships. We could easily give twenty interesting examples of important ways in which the above principle enters into engineering problems. The student may now work out the electrical analogue for himself and study Hertz' vibrations.

149. Steam engine Indicator vibration. The motion of the pencil is to faithfully record the force of

the steam on the piston at every instant; this means that the natural vibrations of the instrument shall be very quickly destroyed by friction. Any friction as of solids on solids will cause errors. Indeed it is easy to see that solid friction causes diagrams to be always larger than they ought to be. Practically we find that if the natural frequency of the instrument is about 20 times that of the engine, the diagram shows few ripples due to the natural vibrations of the indicator. If the natural frequency is only 10 times that of the engine, the diagram is so 'upset' as to be useless.

The frequency of a mass $\frac{W}{g}$ at the end of a spring whose yieldingness is h, see Art. 141, is $\frac{1}{2\pi}\sqrt{g/Wh}$, neglecting friction. We shall consider friction in Art. 160. What is the frequency of a mechanism like what we have in an indicator, controlled by a spring? Answer: If at any point of the indicator mechanism there is a mass $\frac{w}{g}$, and if the displacement of this point is s, when the displacement of the end of the spring (really the piston, in any ordinary indicator) is 1; **imagine that instead of $\frac{w}{g}$ we have a mass $s^2\frac{w}{g}$** at the end of the spring. Thus the frequency is $\frac{1}{2\pi}\sqrt{\frac{g}{h\Sigma s^2 w}}$.

To illustrate this, take the case shown in fig. 80; OAB is a massless lever, hinged at O, with the weight W at B. The massless spring is applied at A.

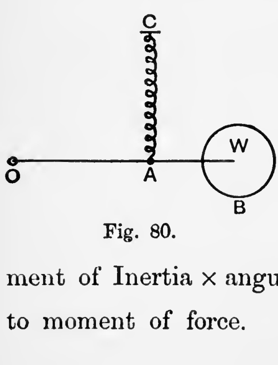

Fig. 80.

When A is displaced downwards from equilibrium through the distance x, the extra pull in the spring is $\frac{x}{h}$. The angular displacement of the lever, clockwise, is $\frac{x}{OA}$. Moment of Inertia × angular acceleration, is numerically equal to moment of force. The Moment of Inertia is $\frac{W}{g}OB^2$.

The angular acceleration is $\frac{\ddot{x}}{OA}$, where $\ddot{x}$ stands for $\frac{d^2x}{dt^2}$, so that $\frac{W}{g} OB^2 \cdot \frac{\ddot{x}}{OA} + \frac{x}{h} \cdot OA = 0$,

or
$$\ddot{x} + \frac{OA^2}{OB^2} \cdot \frac{g}{Wh} \cdot x = 0.$$

And $\frac{OB}{OA}$ is what we called s, so that $s^2 W$ takes the place of our old W when W was hung directly from the spring.

150. Vibration Indicator. Fig. 81 shows an instrument which has been used for indicating quick vertical vibration of the ground.

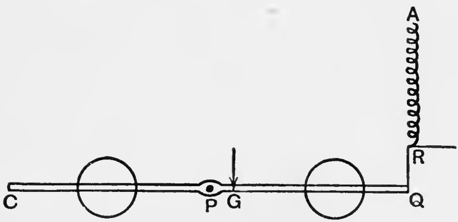

Fig. 81.

The mass CPQ is supported at P by a knife edge, or by friction wheels. The centre of gravity G is in a horizontal line with P and Q. Let $PG = a$, $GQ = b$, $PQ = a + b = l$. The vertical spring AR and thread RQ support the body at Q. As a matter of fact AR is an Ayrton-Perry spring, which shows by the rotation of the pointer R, the relative motion of A and Q; let us neglect its inertia now, and consider that the pointer faithfully records relative motion of A and Q. It would shorten the work to only consider the forces at P and Q in *excess* of what they are when in equilibrium, but for clearness we shall take the total forces.

When a body gets motion in any direction parallel to the plane of the paper, we get one equation by stating that **the resultant force is equal (numerically) to the mass multiplied by the linear acceleration of the centre of gravity in the direction of the resultant force.** We get another equation by stating

that **the resultant moment of force about an axis at right angles to the paper through the centre of gravity is equal to the angular acceleration, multiplied by moment of inertia about this axis through the centre of gravity.** I shall use x, $\dot{x}$ and $\ddot{x}$ to mean displacement, velocity and acceleration, or x, $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$.

Let P and A get a displacement x_1 downward. Let Q be displaced x downward. Let the pull in the spring be $Q = Q_0 + c(x - x_1)$ where c is a known constant (c is the reciprocal of the h used in Art. 141). Let W be the weight of the body. Then if P_0 and Q_0 be the upward forces at the points marked P and Q when in the position of equilibrium,

$$Q_0(a+b) = Wa \text{ and } P_0 + Q_0 = W.$$

Hence
$$P_0 = \frac{bW}{a+b}, \quad Q_0 = \frac{aW}{a+b} \quad \text{..............(1)},$$

$$Q = Q_0 + c(x - x_1).$$

Now G is displaced downwards $\frac{b}{a+b}x_1 + \frac{a}{a+b}x$, so that

$$W - P - Q = \frac{W}{g}\{b\ddot{x}_1 + a\ddot{x}\}\frac{1}{a+b} \quad \text{.........(2)}.$$

The body has an angular displacement θ clockwise about its centre of mass, of the amount $\frac{x - x_1}{a+b}$. So that if I is its moment of inertia about G

$$-Qb + Pa = \frac{I}{a+b}(\ddot{x} - \ddot{x}_1) \text{..............(3)}.$$

Hence (2) and (3) give us, if M stands for $\frac{W}{g}$, and if $I = Mk^2$ where k is the radius of gyration about G,

$$\frac{\ddot{x}}{a+b}\left(\frac{I}{a} + aM\right) + xc\left(\frac{b}{a} + 1\right)$$
$$= \frac{\ddot{x}_1}{a+b}\left(\frac{I}{a} - bM\right) + x_1 c\left(\frac{b}{a} + 1\right) \text{......(4)}.$$

If k_1 is the radius of gyration about P, we find that (4) simplifies to

$$\ddot{x} + n^2x = e^2\ddot{x}_1 + n^2x_1$$

if n stands for $\frac{l}{k_1}\sqrt{\frac{c}{M}} = 2\pi \times$ natural frequency, and e^2 stands for $1 - \frac{al}{k_1^2}$. Call $x - x_1$ by the letter y because it is really y that an observer will note, if the framework and room and observer have the motion x_1. Then as $y = x - x_1$ or $x = y + x_1$

$$\ddot{y} + \ddot{x}_1 + n^2(y + x_1) = e^2\ddot{x}_1 + n^2x_1.$$

So that $$\ddot{y} + n^2y = (e^2 - 1)\,\ddot{x}_1 \quad\text{.....................(5)},$$

or $$\ddot{y} + n^2y + \frac{al}{k_1^2}\ddot{x}_1 = 0 \quad\text{....................(6)}.$$

Let $x_1 = A \sin qt$.

We are neglecting friction for ease in understanding our results, and yet we are assuming that there is enough friction to destroy the natural vibration of the body.

We find that if we assume $y = \alpha \sin qt$, then

$$\alpha = \frac{al}{k_1^2}\frac{q^2}{n^2 - q^2}A.$$

That is, the apparent motion y (and this is what the pointer of an Ayrton-Perry spring will show; or a light mirror may be used to throw a spot of light upon a screen), is $\frac{al}{k_1^2}\frac{q^2}{n^2 - q^2}$ times the actual motion of the framework and room and observer. If q is large compared with n, for example if q is always more than five times n, we may take it that the apparent motion is $\frac{al}{k_1^2}$ times the real motion and is independent of frequency. **Hence any periodic motion whatever** (whose periodic time is less say than $\frac{1}{5}$th of the periodic time of the apparatus) **will be faithfully indicated.**

Note that if $al = k_1^2$ so that Q is what is called the point of percussion, Q is a motionless or 'steady' point. But in practice, the instrument is very much like what is shown in

the figure, and Q is by no means a steady point. Apparatus of the same kind may be used for East and West and also for North and South motions.

151. Any equation containing $\frac{dy}{dx}$ or $\frac{d^2y}{dx^2}$ or any other differential coefficients is said to be a "**Differential Equation.**" It will be found that differential equations contain laws in their most general form.

Thus if x is linear space and t time, the statement $\frac{d^3x}{dt^3}=0$ means that $\frac{d^2x}{dt^2}$, (the acceleration), does not alter. It is the most general expression of uniformly accelerated motion. When we integrate and get $\frac{d^2x}{dt^2}=a$, we have introduced the more definite statement that the constant acceleration is known to be a. When we integrate again and get

$$\frac{dx}{dt}=at+b,$$

we are more definite still, for we say that b is the velocity when $t=0$.

When we integrate again and get

$$x=\tfrac{1}{2}at^2+bt+c,$$

we state that $x=c$ when $t=0$.

Later on, it will be better seen, that many of our great general laws are wrapped up in a simple looking expression in the shape of a differential equation, and it is of enormous importance that when the student sees such an equation he should translate it into ordinary language.

152. An equation like

$$\frac{d^4y}{dx^4}+P\frac{d^3y}{dx^3}+Q\frac{d^2y}{dx^2}+R\frac{dy}{dx}+Sy=X\ldots\ldots\ldots(1),$$

if P, Q, R, S and X are functions of x only, or constants, is said to be a **linear differential equation.**

Most of our work in mechanical and electrical engineering leads to linear equations in which P, Q, &c., are all constant with the exception of X. Thus note (8) and (9) of Art. 146.

Later, we shall see that in certain cases we can find the *complete* solution of (1) when X is 0; that is, that the solution found will include every possible answer. Now suppose this to be $y=f(x)$. We shall see that it will include four arbitrary constants, because $\frac{d^4y}{dx^4}$ is the highest differential coefficient in (1), and we shall prove that if, when X is not 0, we can guess at one solution, and we call it $y=F(x)$, then

$$y=f(x)+F(x)$$

is a solution of (1). We shall find in Chap. III. that this is the *complete* solution of (1).

In the remainder of this chapter we shall only consider P, Q, &c., as constant; let us say

$$\frac{d^4y}{dx^4}+A\frac{d^3y}{dx^3}+B\frac{d^2y}{dx^2}+C\frac{dy}{dx}+Ey=X \quad \text{......(2)},$$

where A, B, C, E are constants and X is a function of x.

We often write (2) in the form

$$\left(\frac{d^4}{dx^4}+A\frac{d^3}{dx^3}+B\frac{d^2}{dx^2}+C\frac{d}{dx}+E\right)y=X \text{......(3)}.$$

153. Taking the very simplest equation like (3). Let

$$\frac{dy}{dx}-ay=0 \text{.........................(4)},$$

it is obvious (see Art. 97) that

$$y=M\epsilon^{ax} \quad \text{.......................(5)}$$

is the solution, where M is any constant whatsoever.

154. Now taking $\frac{d^2y}{dx^2}-a^2y=0$ ....(6),

we see by actual trial that

$$y=M\epsilon^{ax}+N\epsilon^{-ax} \text{.....................(7)}$$

is the solution, where M and N are any constants whatsoever.

But if we take $\frac{d^2y}{dx^2}+n^2y=0$ ....(8),

we see that as the a of (6) is like ni in (8) if i means $\sqrt{-1}$, then

$$y = M\epsilon^{nix} + N\epsilon^{-nix} \quad \text{.................(9)}$$

is the solution of (8). If we try whether this is the case, by differentiation, assuming that i behaves like a real quantity and of course $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, &c., we find that it is so. But what meaning are we to attach to such an answer as (9)? By guessing and probably also through recollection of curious analogies such as we describe in Art. 106, and by trial, we find that this is the complete solution also,

$$y = M_1 \sin nx + N_1 \cos nx \quad \text{..............(10).}$$

As (10) and (9) are both *complete* solutions (Art. 152) because they both contain two arbitrary constants which may be unreal or not, we always consider an answer like (9) to be the same as (10), and the student will find it an excellent exercise to convert the form (10) into the form (9) by the exponential forms of $\sin ax$ and $\cos ax$, Art. 106, recollecting that the arbitrary constants may be real or unreal. Besides, it is important for the engineer to make a practical use of those quantities which the mathematicians have called *unreal*.

155. Going back now to the more general form (3) **when $X = 0$,** we try if $y = M\epsilon^{mx}$ is a solution, and we see that it is so if

$$m^4 + Am^3 + Bm^2 + Cm + E = 0 \quad \text{...........(1).}$$

This is usually called the *auxiliary* equation. Find the four roots of it, that is, the four values of m which satisfy it, and if these are called m_1, m_2, m_3, m_4, we have

$$y = M_1\epsilon^{m_1x} + M_2\epsilon^{m_2x} + M_3\epsilon^{m_3x} + M_4\epsilon^{m_4x}$$

as the complete solution of (3) when $X = 0$; M_1, &c., being any arbitrary constants whatsoever.

156. Thus to solve

$$\frac{d^4y}{dx^4} + 5\frac{d^3y}{dx^3} + 5\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 6y = 0,$$

if we assume $y = \epsilon^{mx}$, we find that m must satisfy

$$m^4 + 5m^3 + 5m^2 - 5m - 6 = 0.$$

By guessing we find that $m=1$ is a root; dividing by $m-1$ and again guessing, we find that $m=-1$ is a root; again dividing by $m+1$ we are left with a quadratic expression, and we soon see that $m=-2$ and $m=-3$ are the remaining roots. Hence

$$y = M_1\epsilon^{x} + M_2\epsilon^{-x} + M_3\epsilon^{-2x} + M_4\epsilon^{-3x}$$

is the complete solution, M_1, M_2, &c., being any constants whatsoever.

157. Now an equation like (1) may have an unreal root like $m+ni$, where i is written for $\sqrt{-1}$, and if so, we know from algebra, that these unreal roots go in pairs; when there is one like $m+ni$ there is another like $m-ni$. The corresponding answers for y are

$$y = M_1\epsilon^{(m-ni)x} + N_1\epsilon^{(m+ni)x},$$

or

$$\epsilon^{mx}\{M_1\epsilon^{-nix} + N_1\epsilon^{+nix}\},$$

and we see from (10) that this may be written

$$y = \epsilon^{mx}\{M\sin nx + N\cos nx\},$$

where M and N are any constants whatsoever.

158. Suppose that two roots m of the auxiliary equation, happen to be equal, there is no use in writing

$$y = M_1\epsilon^{mx} + M_2\epsilon^{mx},$$

because this only amounts to $(M_1+M_2)\,\epsilon^{mx}$ or $M\epsilon^{mx}$ where M is an arbitrary constant, whereas the general answer must have two arbitrary constants. In this case we adopt an artifice; we assume that the two roots are m and $m+h$ and we imagine h to get smaller and smaller without limit:

$$y = M_1\epsilon^{mx} + M_2\epsilon^{(m+h)x}$$
$$= \epsilon^{mx}(M_1 + M_2\epsilon^{hx}),$$

but by Art. 97, $\epsilon^{hx} = 1 + hx + \frac{h^2x^2}{2} + \frac{h^3x^3}{1.2.3} + \&c.,$

therefore $y = \epsilon^{mx}\left(M_1 + M_2 + M_2hx + M_2\frac{h^2x^2}{2} + \&c.\right).$

Now let M_2h be called N and imagine h to get smaller

and smaller, and M_2 to get larger and larger, so that M_2h may be of any required value we please, say N, and also

$$M_1 + M_2 = M;$$

as h gets smaller and smaller without limit we find

$$y = \epsilon^{mx}(M + Nx).$$

If this reasoning does not satisfy the reader, he is to remember that we can test our answer and we always find it to be correct.

159. It is in this way that we are led **to the following general rule** for the solving of a linear differential equation with constant coefficients. Let the equation be

$$\frac{d^n y}{dx^n} + A\frac{d^{n-1} y}{dx^{n-1}} + B\frac{d^{n-2} y}{dx^{n-2}} + \&c. + G\frac{dy}{dx} + Hy = 0 \ldots (1).$$

Form the auxiliary equation

$$m^n + Am^{n-1} + Bm^{n-2} + \&c. + Gm + H = 0.$$

The complete value of y will be expressed by a series of terms:—For each real distinct value of m, call it α_1, there will exist a term $M_1\epsilon^{\alpha_1 x}$; for each pair of imaginary values $\alpha_2 \pm \beta_2 i$, a term

$$\epsilon^{\alpha_2 x}(M_2 \sin \beta_2 x + N_2 \cos \beta_2 x);$$

each of the coefficients M_1, M_2, N_2 being an arbitrary constant if the corresponding root occurs only once, but a polynomial of the $r-1$th degree with arbitrary constant coefficients if the root occur r times.

Exercise.
$$\frac{d^5y}{dx^5} + 12\frac{d^4y}{dx^4} + 66\frac{d^3y}{dx^3} + 206\frac{d^2y}{dx^2} + 345\frac{dy}{dx} + 234y = 0.$$

Forming the auxiliary equation, I find by guessing and trying, that the five roots are

$$-3, -3, -2, -2+3i, -2-3i.$$

Consequently the answer is

$$y = (M_1 + N_1x)\epsilon^{-3x} + M_2\epsilon^{-2x} + \epsilon^{-2x}(M_3 \sin 3x + N_3 \cos 3x).$$

Exercise. 1. Integrate $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0.$

Answer: $y = A\epsilon^{3x} + B\epsilon^{x}$.

2. Integrate $\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 34y = 0.$

Answer: $y = \epsilon^{5x}\{A \sin 3x + B \cos 3x\}$.

3. Integrate $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0.$

Answer: $y = (A + Bx)\,\epsilon^{-3x}$.

4. Integrate $\frac{d^4y}{dx^4} - 12\frac{d^3y}{dx^3} + 62\frac{d^2y}{dx^2} - 156\frac{dy}{dx} + 169y = 0.$

Here $m^4 - 12m^3 + 62m^2 - 156m + 169 = 0$, and this will be found to be a perfect square. The roots of the auxiliary equation will be found to be

$$3 + 2i,\ 3 + 2i,\ 3 - 2i,\ 3 - 2i.$$

Hence the solution is

$$y = \epsilon^{3x}\{(A_1 + B_1x)\sin 2x + (A_2 + B_2x)\cos 2x\}.$$

We shall now take an example which has an important physical meaning.

Natural Vibrations. Example.

160. We had in Art. 146, a **mechanical** system vibrating with one degree of freedom, and we saw that it was analogous with the surging going on in an **Electric** system consisting of a condenser, and a coil with resistance and self-induction. We neglected the friction in the mechanical, and the resistance in the electric problem. We shall now study their natural vibrations, and we choose the mechanical problem as before. If a weight of W lb. hung at the end of a spring which elongates x feet for a force of $x \div h$ lb., is resisted in its motion by friction equal to $b \times$ velocity, then we had (8) of Art. 146, or

$$\frac{W}{g}\frac{d^2x}{dt^2} + b\frac{dx}{dt} + \frac{x}{h} = 0,$$

or

$$\frac{d^2x}{dt^2} + \frac{bg}{W}\cdot\frac{dx}{dt} + \frac{xg}{Wh} = 0 \quad\ldots\ldots\ldots\ldots\ldots(1).$$

Let $\frac{bg}{W}$ be called $2f$ and let $\frac{g}{Wh} = n^2$; (1) becomes

$$\frac{d^2x}{dt^2} + 2f \cdot \frac{dx}{dt} + n^2x = 0 \quad \dots\dots\dots\dots\dots(2).$$

Forming the auxiliary equation we find the roots to be

$$m = -f \pm \sqrt{f^2 - n^2}.$$

We have different kinds of answers depending upon the values of f and n. We must be given sufficient information about the motion to be able to calculate the arbitrary constants. I will assume that when t is 0 the body is at $x = 0$ and is moving with the velocity v_0.

I. Let f be greater than n, and let the roots be $-\alpha$ and $-\beta$.

II. Let f be equal to n, the roots are $-f$ and $-f$.

III. Let f be less than n, and let the roots be $-a \pm bi$.

IV. Let $f = 0$, the roots are $\pm ni$.

Then according to our rule of Art. 159,

In Case I, our answer is

$$x = A\epsilon^{-\alpha t} + B\epsilon^{-\beta t},$$

and if we are told that $x = 0$ when $t = 0$ and $\frac{dx}{dt} = v_0$ when $t = 0$, we can calculate A and B and so find x exactly in terms of t;

In Case II, our answer is

$$x = (A + Bt)\,\epsilon^{-ft};$$

In Case III, our answer is

$$x = \epsilon^{-at}\{A \sin bt + B \cos bt\};$$

In Case IV, our answer is

$$x = A \sin nt + B \cos nt.$$

161. We had better take a numerical example and we assure the student that he need not grudge any time spent upon it and others like it. Let $n = 3$ and take various values of f. For the purpose of comparison we shall in all cases let $x = 0$ when $t = 0$; and $\frac{dx}{dt} = 20$ feet per second, when $t = 0$.

Case IV. Let $f=0$, then $x=A\sin nt+B\cos nt$,

$$0=A\times 0+B\times 1,\ \text{so that } B=0,$$

$$\frac{dx}{dt}=nA\cos nt-nB\sin nt,$$

$$20=3A,\ \text{so that } A=\tfrac{20}{3}.$$

Plot therefore $x=6{\cdot}667\sin 3t$.

This is shown in curve 4, fig. 82. It is of course the ordinary curve of sines: undamped S.H. motion.

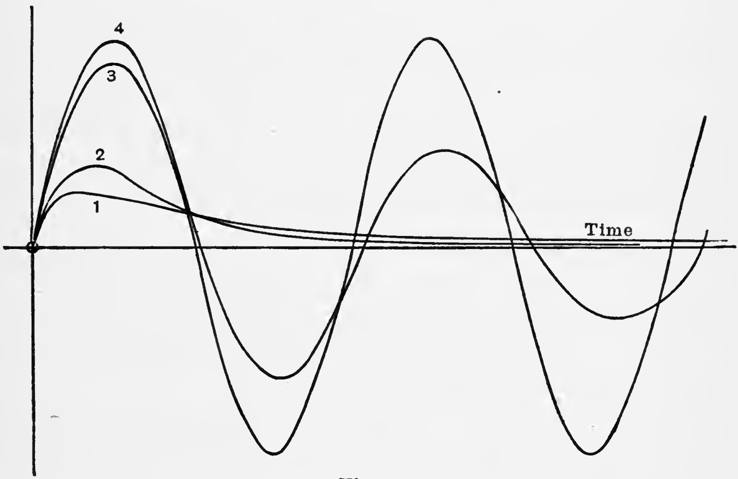

Fig. 82.

Case III. Let $f=\cdot 3$. The auxiliary equation gives

$$m=-\cdot 3\pm\sqrt{\cdot 09-9}=-\cdot 3\pm 2{\cdot}985i.$$

Here $a=\cdot 3$ and $b=2{\cdot}985$ in

$$x=\epsilon^{-at}\{A\sin bt+B\cos bt\}\ \ldots\ldots\ldots\ldots(1).$$

You may not be able to differentiate a product yet, although we gave the rule in Art. 90. We give many exercises in Chap. III. and we shall here assume that

$$\frac{dx}{dt}=-a\epsilon^{-at}(A\sin bt+B\cos bt)$$
$$+b\epsilon^{-at}(A\cos bt-B\sin bt)\ \ldots\ldots\ldots\ldots(2).$$

Put $x=0$ when $t=0$ and $\frac{dx}{dt}=20$ when $t=0$. Then $B=0$ from (1) and

$$20=bA \text{ or } A=\frac{20}{b}=\frac{20}{2\cdot985}=6\cdot7,$$

and hence $$x=6\cdot7\epsilon^{-\cdot3t}\sin 2\cdot985t.$$

This is shown in curve 3 of fig. 82. Notice that the period has altered because of friction.

Case II. Let $f=3$. The roots of the auxiliary equation are $m=-3$ and -3, equal roots. Hence

$$x=(A+Bt)\,\epsilon^{-3t}\,\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Here again we have to differentiate a product and

$$\frac{dx}{dt}=B\epsilon^{-3t}-3\,(A+Bt)\,\epsilon^{-3t}\,\ldots\ldots\ldots\ldots(2).$$

Putting in $x=0$ when $t=0$ and $\frac{dx}{dt}=20$ when $t=0$, $A=0$ from (1) and $B=20$ from (2).

Hence $$x=20t\,.\,\epsilon^{-3t}.$$

This is shown in curve 2 of fig. 82.

Case I. Let $f=5$. The roots of the auxiliary equation are -9 and -1,

$$x=A\epsilon^{-9t}+B\epsilon^{-t},$$

$$\frac{dx}{dt}=-9A\epsilon^{-9t}-B\epsilon^{-t}.$$

Putting in the initial conditions we have

$$0=A+B,\quad 20=-9A-B.$$

Hence $$A=-2\tfrac{1}{2},\quad B=2\tfrac{1}{2},$$

$$x=2\tfrac{1}{2}\,(\epsilon^{-t}-\epsilon^{-9t}).$$

This is shown in curve 1 of fig. 82.

Students ought to take these initial conditions

$$x=10 \text{ when } t=0 \text{ and } \frac{dx}{dt}=0 \text{ when } t=0.$$

This would represent the case of a body let go at time 0 or, in the electrical case, a charged condenser begins to be discharged at time 0.

Notice that if we differentiate (1), Art. 160, all across we have $\left(\text{using } v \text{ for } \frac{dx}{dt}\right)$,

$$\frac{d^2v}{dt^2} + \frac{bg}{W}\frac{dv}{dt} + \frac{g}{Wh}\cdot v = 0.$$

We have therefore exactly the same law for velocity or acceleration that we have for x itself.

Again, in the electrical case as $K\frac{dv}{dt}$ represents current, if we differentiate all across we find exactly the same law for current as for voltage. Of course differences are produced in the solutions of the equations by the initial conditions.

162. When the right-hand side of such a linear differential equation as (2) Art. 152 is not zero and our solution will give the forced motion of a system as well as the natural vibrations, it is worth while to consider the problem from a point of view which will be illustrated in the following simple example.

To solve (11) Art. 148, which is

$$\frac{d^2x}{dt^2} + n^2x = n^2a \sin qt \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

the equation of motion of a system with one degree of freedom and without friction.

Differentiate twice and we find

$$\frac{d^4x}{dt^4} + n^2\frac{d^2x}{dt^2} = -n^2q^2a \sin qt.$$

Hence from (1), $\frac{d^4x}{dt^4} + (n^2+q^2)\frac{d^2x}{dt^2} + q^2n^2x = 0 \ldots\ldots\ldots\ldots(2).$

To solve (2), the auxiliary equation is

$$m^4 + (n^2+q^2)m^2 + q^2n^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

and we know that $\pm ni$ are two roots and $\pm qi$ are the other two roots. Hence we have the complete solution

$$x = A \sin nt + B \cos nt + C \sin qt + D \cos qt \ldots\ldots(4).$$

Now it was by differentiating (1) that we introduced the possibility of having the two extra arbitrary constants C and D, and evidently by inserting (4) in the original equation, we shall find the proper values of C and D, as they are really not arbitrary. It will be noticed that by differentiating (1) and obtaining (2) **we made the system more complex,** gave it another degree of freedom, or rather we made it part of a larger system, a system whose natural vibrations are given in (4). When we let a mass vibrate at the end of a spring, it is to be remembered that the centre of gravity of the mass and the frame which supports it and the room, remain unaltered. Hence vibrations occur in the supporting frame, and there is friction tending to still the vibrations. If there is another mass also vibrating, this effect may be lessened. For example in fig. 83, if M vibrates at the end of the strip MA, clamped in the vice A, any motion of M to the right must be accompanied by motion of A and the support, to the left. But if we have two masses M_1 and M_2 (as in a tuning fork), moving in opposite directions at each instant there need be no motion of the supports, consequently the system M_1M_2 vibrates as if there were less friction, and this principle is utilized in tuning forks. Should a motion be started, different from this, it will quickly become like this, **as any part of the motion which necessitates a motion of the centre of gravity of the supports, is very quickly damped out of existence.** The makers of steam engines and the persons who use them in cities where vibration of the ground is objected to, find it important to take matters like this into account.

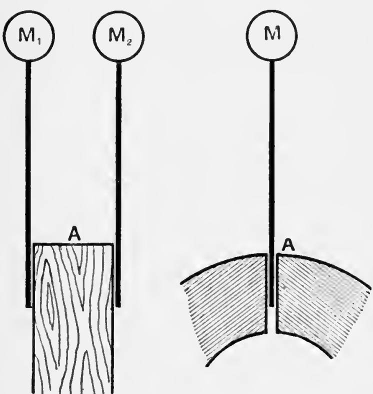

Fig. 83.

163. If y is a known function of x, we are instructed by

(3), Art. 152, to perform a complicated operation upon it. Sometimes we use such a symbol as

$$(\theta^4 + A\theta^3 + B\theta^2 + C\theta + E)\, y = X,$$

to mean exactly the same thing; θy meaning that we differentiate y with regard to x, $\theta^2 y$ meaning that we differentiate y twice, and so on.

θ, θ^2, &c., are **symbols of operation** easy enough to understand. We need hardly say that $\theta^2 y$ does not mean that there is a quantity θ which is squared and multiplied upon y: it is merely a convenient way of saying that y is to be differentiated twice. $\theta\theta y$ would mean the same thing. On this same system, what does $(\theta + a)\, y$ mean? It means $\frac{dy}{dx} + ay$. What does $(\theta^2 + A\theta + B)\, y$ mean? It means $\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By$. $(\theta + a)\, y$ instructs us to differentiate y and add a times y, for a is a mere multiplier although θ is not so, and yet, note that $(\theta + a)\, y = \theta y + ay$.

In fact we find that θ enters into these operational expressions as if it were an algebraic quantity, although it is not one.

If u and v are functions of x we know that

$$\theta\,(u + v) = \theta u + \theta v.$$

This is what is called the **distributive law.**

Again, if a is a constant, $\theta au = a\theta u$, or the operation θa is equivalent to the operation $a\theta$. This is called the **commutative law.**

Again $\theta^n \theta^m = \theta^{m+n}$; this is the **index law.** When these three laws are satisfied we know that θ will enter into ordinary algebraic expressions as if it were a quantity. θ follows all these laws when combined with constants; but note that if u and v are functions of x, $v\theta u$ meaning $v\frac{du}{dx}$, is a very different thing from $\theta\,.uv$. When we are confining our

attention to linear operations we are not likely to make mistakes.

Thus operate with $\theta + b$ upon $(\theta + a)\,y$. Now

$$(\theta + a)\,y = \theta y + ay \text{ or } \frac{dy}{dx} + ay.$$

Operating with $\theta + b$ means "differentiate (this gives us $\frac{d^2y}{dx^2} + a\frac{dy}{dx}\Big)$ and add b times $\frac{dy}{dx} + ay$." Consequently it gives us $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + b\frac{dy}{dx} + aby$ or $\frac{d^2y}{dx^2} + (a+b)\frac{dy}{dx} + aby$ or

$$\{\theta^2 + (a+b)\,\theta + ab\}\,y.$$

We see, therefore, that the double operation

$$(\theta + b)\,(\theta + a)$$

gives the same result as

$$\{\theta^2 + (a+b)\,\theta + ab\}.$$

In this and other ways it is easy to show that although θ is a symbol of operation and not a quantity, yet it enters into combinations as if it were an algebraic quantity, so long as all the quantities a, b, &c. are constants. Note also that

$$(\theta + a)\,(\theta + b)$$

is the same as $\quad (\theta + b)\,(\theta + a).$

The student ought to practise and see that this is so and get familiar with this way of writing. He will find that it saves an enormous amount of unnecessary trouble. Thus compare such expressions as

$$(a\theta + b)\,(\alpha\theta + \beta)\,y$$

with $\quad \{a\alpha\theta^2 + (a\beta + \alpha b)\,\theta + b\beta\}\,y,$

or $\quad a\alpha\frac{d^2y}{dx^2} + (a\beta + \alpha b)\frac{dy}{dx} + b\beta y.$

164. Suppose that Dy is used as a symbol for some curious operation to be performed upon y, and we say that $Dy = X$; does this not mean that if we only knew how to reverse the

operation, and we indicate the reverse operation by D^{-1} or $\frac{1}{D}$, then $y = D^{-1}X$ or $\frac{X}{D}$? We evidently mean that if we operate with D upon $D^{-1}X$, we annul the effect of the D^{-1} operation. Now if $\frac{dy}{dx} + ay = X$, or $\left(\frac{d}{dx} + a\right) y = X$, or $(\theta + a)\, y = X$, let us indicate the reverse operation by

$$y = \left(\frac{d}{dx} + a\right)^{-1} X \text{ or } (\theta + a)^{-1} X \quad \text{.........(1)},$$

or

$$\frac{X}{\frac{d}{dx} + a} \text{ or } \frac{X}{\theta + a} \quad \text{.....................(2)}.$$

Keeping to the last of these; at present $\frac{1}{\theta + a}$ is a mere symbol for an inverse operation, but

$$y = \frac{X}{\theta + a} \quad \text{...........................(3)}$$

submits to the usual rules of multiplication, because (3) is the same as $(\theta + a)\, y = X$ ....(4);

and yet (4) is derived from (3) *as if* by the multiplication of both sides of the equation by $(\theta + a)$.

Again, take $\frac{d^2y}{dx^2} + (a + b)\frac{dy}{dx} + aby = X$................(5),

or $\{\theta^2 + (a + b)\,\theta + ab\}\, y = X$ ....(6),

or $(\theta + a)(\theta + b)\, y = X$ ....(7).

Here the direct operation $\theta + a$ performed upon $(\theta + b)\, y$ gives us X; hence by the above definition

$$(\theta + b)\, y = \frac{X}{\theta + a} \quad \text{.......................(8)},$$

and repeating, we have

$$y = \frac{1}{(\theta + b)}\left(\frac{X}{\theta + a}\right) \quad \text{.....................(9)}.$$

But it is consistent with our way of writing inverse operations to write (6) as

$$y = \frac{X}{\theta^2 + (a+b)\theta + ab} \quad \text{.................(10)},$$

and so we see that there is nothing inconsistent in our treating the $\theta + b$ and $\theta + a$ of (9) as if θ were an algebraic quantity.

165. We know now that the inverse operation

$$\{\theta^2 + (a+b)\theta + ab\}^{-1} \quad \text{.................(1)},$$

may be effected in two steps; first operate with $(\theta + b)^{-1}$ and then operate with $(\theta + a)^{-1}$.

Here is a most interesting question. We know that if θ were really an algebraic quantity,

$$\frac{1}{\theta^2 + (a+b)\theta + ab} = \frac{1}{b-a}\left(\frac{1}{\theta + a} - \frac{1}{\theta + b}\right) \quad \text{......(2)}.$$

And it is important to know if the operation

$$\frac{\mathbf{1}}{\mathbf{b-a}}\left(\frac{\mathbf{1}}{\boldsymbol{\theta}+\mathbf{a}} - \frac{\mathbf{1}}{\boldsymbol{\theta}+\mathbf{b}}\right) \quad \text{...................(3)},$$

is exactly the inverse of $\boldsymbol{\theta}^2 + (\mathbf{a}+\mathbf{b})\,\boldsymbol{\theta} + \mathbf{ab}$?...(4). Our only test is this; it is so, if the direct operation (4) completely annuls (3). Apply (3) to X and now apply (4) to the result; if we apply (4) to $\frac{1}{\theta + a}X$, we evidently obtain $(\theta + b)X$ or $\frac{dX}{dx} + bX$; if we apply (4) to $\frac{1}{\theta + b}X$ we evidently obtain $(\theta + a)X$ or $\frac{dX}{dx} + aX$, and

$$\frac{1}{b-a}\left\{\frac{dX}{dx} + bX - \left(\frac{dX}{dx} + aX\right)\right\} = X.$$

We see therefore that (3) is the inverse of (4), and that we have the right to split up an inverse operation like the left-hand side of (2) into partial operations like the right-hand side of (2). We have already had a number of illustrations of this when the operand was 0. For it is obvious that if α_1, α_2, &c. are the roots of the auxiliary equation of Art. 159, it really means that

$$\theta^n + A\theta^{n-1} + B\theta^{n-2} + \text{\&c.} + G\theta + H$$

splits up into the factors $(\theta - \alpha_1)(\theta - \alpha_2)$, &c.

Observe that if $\frac{dy}{dx}=X$, or $\theta y=X$, or $y=\frac{X}{\theta}$, or $y=\theta^{-1}X$, the inverse operation θ^{-1} simply means that X is to be integrated. Again, θ^{-2} means integrate twice, and so on*.

---

* Suppose in our operations we ever meet with the symbols $\theta^{\frac{1}{2}}$ or $\theta^{-\frac{1}{2}}$ or $\theta^{\frac{3}{2}}$ &c., what interpretations are we to put upon them? It is not very necessary to consider them now. Whatever interpretations we may put upon them must be consistent with everything we have already done. For example $\theta^{\frac{3}{2}}$ will be the same as $\theta\theta^{\frac{1}{2}}$ and $\theta^{-\frac{1}{2}}$ will be the same as $\theta^{\frac{1}{2}}\theta^{-1}$ or $\theta^{-1}\theta^{\frac{1}{2}}$. We have to recollect that all this work is *integration* and we use symbols to help us to find answers; we are employing a scientific method of guessing, and our great test of the legitimacy of a method is to try if our answer is right; this can always be done. Most of the functions on which we shall be operating are either of the shape $A\epsilon^{ax}$ or $B\sin bx$ or sums of such functions. Observe that

$$\theta^n A\epsilon^{ax}=Aa^n\epsilon^{ax},$$

if n is an integer either positive or negative. There is therefore a likelihood that it will help in the solution of problems to assume that

$$\theta^{\frac{1}{2}}A\epsilon^{ax}=Aa^{\frac{1}{2}}\epsilon^{ax},$$

or that

$$\boldsymbol{\theta^{\frac{1}{3}}A\epsilon^{ax}=Aa^{\frac{1}{3}}\epsilon^{ax}} \quad\text{.............................(1).}$$

Again

$$\theta B\sin bx=Bb\cos bx=Bb\sin\left(bx+\frac{\pi}{2}\right),$$

$$\theta^2 B\sin bx=-Bb^2\sin bx=Bb^2\sin(bx+\pi),$$

and

$$\theta^n B\sin bx=Bb^n\sin\left(bx+n\frac{\pi}{2}\right) \quad\text{.......................(2).}$$

Evidently this is true when n is a positive or negative integer; assume it true when n is a positive or negative fraction, so that

$$\boldsymbol{\theta^{\frac{1}{2}}B\sin bx=Bb^{\frac{1}{2}}\sin\left(bx+\frac{\pi}{4}\right)} \quad\text{...................(3).}$$

There are certain other useful functions as well as ϵ^{ax} and *sin* bx such that we are able to give a meaning to the effect of operating with $\theta^{\frac{1}{2}}$ upon them. It will, for example, be found, if we pursue our subject, that we shall make use of a function which is 0 for all negative values of x and which is a constant a for all positive values of x. It will be found that if this function is called $f(x)$ then

$$\boldsymbol{\theta^{\frac{1}{2}}f(x)=a\frac{1}{\sqrt{\pi}}x^{-\frac{1}{2}}} \quad\text{...............................(4),}$$

and the meaning of $\theta^{\frac{3}{2}}$ or $\theta^{\frac{5}{2}}$ or $\theta^{-\frac{1}{2}}$ or $\theta^{-\frac{3}{2}}$ &c. is easily obtained by differentiation or integration. The Mnemonic for this, we need not call it proof or reason, is $\theta^n x^m=\frac{\lfloor m}{\lfloor m-n}x^{m-n}$. Let $n=\frac{1}{2}$, $m=0$ and we have

166. Electrical Problems. Circuit with resistance R and self-induction L,

$$V = RC + L\frac{dC}{dt};$$

let $\frac{d}{dt}$ be indicated by θ, then

$$V = (R + L\theta)\,C \text{ or } C = \frac{V}{R + L\theta}.$$

In fact in all our algebraic work we treat $\mathbf{R + L\theta}$ as if it were a resistance.

Condenser of capacity K farads. Let V volts, be voltage between coatings. Let C be *current* in amperes *into* the condenser, that is, the rate at which Q, its charge in coulombs, is increasing. Or $C = \frac{dQ}{dt} = \frac{d}{dt}(KV)$ or as K is usually assumed to be constant, $C = K\frac{dV}{dt}$.

The conductance of a condenser is $K\theta$, therefore

$$C = K\,.\,\theta\,.\,V = V \div \frac{1}{K\theta}.$$

Hence the current *into* a condenser is as if the condenser had a resistance $\frac{\mathbf{1}}{\mathbf{K\theta}}$.

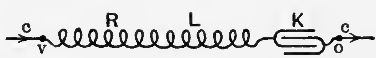

Fig. 84.

Circuit with resistance, self-induction and capacity, fig. 84. All problems are worked out as if we had a total resistance

$$\mathbf{R + L\theta + \frac{1}{K\theta}} \quad \text{.....................(1).}$$

$\theta^{\frac{1}{2}}x^0 = \frac{1}{\lfloor\underline{-\frac{1}{2}}} x^{-\frac{1}{2}}$. But $\lfloor\underline{-\frac{1}{2}}$ has no meaning. Give it a meaning by assuming that what is true of integers, is true of all numbers, and use gamma function of $\frac{1}{2}$ or $\Gamma\frac{1}{2}$ which is $\sqrt{\pi}$ instead of $\lfloor\underline{-\frac{1}{2}}$. It is found that the solutions effected by means of this are correct.

167. In any **network of conductors** we can say exactly what is the actual resistance (for steady currents) between any point A and another point B if we know all the resistances r_1, r_2, &c. of all the branches. Now if each of these branches has self-induction l_1, &c. and capacity K_1, &c. what we have to do is to substitute $r_1 + l_1\theta + \dfrac{1}{K_1\theta}$ instead of r_1 in the mathematical expressions, and we have the resistance right for currents that are not steady.

How are we to understand our results? However complicated an operation we may be led to, when cleared of fractions, &c. it simplifies to this; that an operation like

$$\frac{a + b\theta + c\theta^2 + d\theta^3 + e\theta^4 + f\theta^5 + \&c}{a' + b'\theta + c'\theta^2 + d'\theta^3 + e'\theta^4 + f'\theta^5 + \&c.} \quad \ldots\ldots(1),$$

has to be performed upon some voltage which is a function of the time. On some functions of the time which we have studied we know the sort of answer which we shall obtain. Thus notice that if we perform $(a + b\theta + \&c.)$ upon $\epsilon^{\alpha t}$ we obtain

$$(a + b\alpha + c\alpha^2 + d\alpha^3 + e\alpha^4 + f\alpha^5 + \&c.)\,\epsilon^{\alpha t} \quad \ldots\ldots(2).$$

Consequently the complicated operation (1) comes to be a mere multiplication by A and division by A', where A is the number $a + b\alpha + c\alpha^2 + \&c.$ and A' is the number

$$a' + b'\alpha + c'\alpha^2 + \&c.$$

Again, if we operate upon $m \sin(nt + \epsilon)$, observe that

$$\begin{array}{lll} & \theta^2 \text{ would give} & -mn^2 \sin(nt+\epsilon), \\ \text{and} & \theta^4 \quad ,, \quad ,, & +mn^4 \sin(nt+\epsilon), \\ \text{and so on;} & & \\ \text{whereas} & \theta \text{ would give} & mn \cos(nt+\epsilon), \\ & \theta^3 \quad ,, \quad ,, & -mn^3 \cos(nt+\epsilon), \\ & \theta^5 \quad ,, \quad ,, & +mn^5 \cos(nt+\epsilon). \end{array}$$

And hence the complicated operation (1) produces the same effect as $\dfrac{p + q\theta}{\alpha + \beta\theta}$, where

$$p = a - cn^2 + en^4 - \&c., \qquad q = b - dn^2 + fn^4 - \&c.$$
$$\alpha = a' - c'n^2 + e'n^4 - \&c., \qquad \beta = b' - d'n^2 + f'n^4 - \&c.$$

Observe Art. 118, that $p+q\theta$ operating upon $m\sin(nt+\epsilon)$ multiplies the amplitude by $\sqrt{p^2+q^2n^2}$ and causes an *advance* of $\tan^{-1}\frac{qn}{p}$. The student ought to try this again for himself, although he has already done it in another way. Show that

$$(p+q\theta)\sin nt = \sqrt{p^2+q^2n^2}\sin\left(nt+\tan^{-1}\frac{qn}{p}\right).$$

Similarly, the inverse operation $1/(\alpha+\beta\theta)$ divides the amplitude by $\sqrt{\alpha^2+\beta^2n^2}$ and produces a *lag* of $\tan^{-1}\frac{\beta n}{\alpha}$, and hence

$$\frac{p+q\theta}{\alpha+\beta\theta}m\sin(nt+\epsilon)$$
$$=m\sqrt{\frac{p^2+q^2n^2}{\alpha^2+\beta^2n^2}}\sin\left(nt+\epsilon+\tan^{-1}\frac{qn}{p}-\tan^{-1}\frac{\beta n}{\alpha}\right),$$

a labour-saving rule of enormous importance.

168. In all this we are thinking only of the **forced vibrations** of a system. We have already noticed that when we have an equation like (1) or (2) Art. 152, the solution consists of two parts, say $y=f(x)+F(x)$; where $f(x)$ is the answer if X of (2) is 0, the natural action of the system left to itself, and $F(x)$ is the forced action. If in (2) we indicate the operation

$$\left(\frac{d^4}{dx^4}+A\frac{d^3}{dx^3}+B\frac{d^2}{dx^2}+C\frac{d}{dx}+E\right)y \text{ by } Dy,$$

then $D(y)=X$ gives us

$$y=D^{-1}(0)+D^{-1}(X).$$

Where $D^{-1}(0)$ gives $f(x)$ and $D^{-1}(X)$ gives $F(x)$.

Thus if $\frac{dy}{dx}+ay=0$, or $\left(\frac{d}{dx}+a\right)y=0$, or $(\theta+a)y=0$, we know Art. 97, that $y=A\epsilon^{-ax}$.

Hence we see that $\boldsymbol{\frac{0}{\theta+a}}$ **is not nothing,** but is $A\epsilon^{-ax}$, so that if $\frac{dy}{dx}+ay=X$, the complete solution is

$$y=A\epsilon^{-ax}+\frac{X}{\theta+a}.$$

We are now studying this latter part, the forced part, only. In most practical engineering problems the exponential terms rapidly disappear.

169. Thus in an electric circuit where $V=(R+L\theta)\,C$, if

$$\mathbf{V = V_0 \sin qt},$$

we have already found the forced value of C,

$$C=\frac{V_0 \sin qt}{R+L\theta},$$

and according to our new rule, or according to Art. 118, this becomes

$$C=\frac{V_0}{\sqrt{R^2+L^2q^2}}\sin\left(qt-\tan^{-1}\frac{Lq}{R}\right) \text{ ....(1).}$$

But besides this term we have one

$$=\frac{0}{R+L\theta} \text{ or } \frac{0}{\theta+\frac{R}{L}},$$

and according to the above rule (Art. 168) this gives a term

$$A_1\epsilon^{-\frac{R}{L}t} \text{ ....(2).}$$

Or we may get this term as in Art. 97,

$$RC+L\frac{dC}{dt}=0, \text{ or } \frac{dC}{dt}=-\frac{R}{L}C.$$

This is the compound interest law and gives us the answer (2), and the sum of (2) and (1) is the complete answer. If we know the value of C when $t=0$, we can find the value of the constant A_1; (2) is obviously an evanescent term.

Thus again, suppose **V to be constant = $\mathbf{V_0}$,**

$$C=\frac{V_0}{R+L\theta}.$$

It is evident that $C=\frac{V_0}{R}$ is the forced current, for if we operate on $C=\frac{V_0}{R}$ with $R+L\theta$ we obtain V_0, and the evanes-

cent current is always the same with the same R and L whatever V may be, namely $A_1\epsilon^{-\frac{R}{L}t}$.

The complete answer is then

$$C = A_1\epsilon^{-\frac{R}{L}t} + \frac{V_0}{R} \quad \text{..........................(2).}$$

Let, for example, $C=0$ when $t=0$, then

$$0 = A_1 + \frac{V_0}{R} \text{ or } A_1 = -\frac{V_0}{R},$$

and (2) becomes $\quad C = \frac{V_0}{R}\left(1 - \epsilon^{-\frac{R}{L}t}\right)$ ....(3).

The student ought to take $V_0 = 100$, $R=1$, $L=\cdot 1$ and show how C increases. We have had this law before.

170. *Example.* **A condenser of capacity K and a non-inductive resistance r in parallel;** voltage V at their terminals, fig. 85. The two currents are $c = V/r$, $C = K\theta V$, and their sum is $C+c = V\left(\frac{1}{r} + K\theta\right)$ or $V\left(\frac{1+rK\theta}{r}\right)$, so that the two in parallel act like a resistance $\frac{r}{1+rK\theta}$.

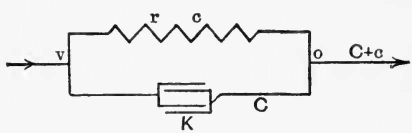

Fig. 85.

If $V = V_0 \sin nt$,

$$C + c = \frac{(1+rK\theta)\,V_0 \sin nt}{r}, \text{ and by Art. 167,}$$

$$C + c = \frac{V_0}{r}\sqrt{1+r^2K^2n^2}\,.\sin\left(nt + \tan^{-1} rKn\right),$$

$$c = \frac{V_0}{r}\sin nt, \quad C = V_0Kn \sin\left(nt + \frac{\pi}{2}\right).$$

171. A circuit with resistance, self-induction and capacity (fig. 86) has the **alternating voltage** $V = V_0 \sin nt$ established at its ends; what is the current?

Answer, $C = \dfrac{V}{R + L\theta + \dfrac{1}{K\theta}}$, and by Art. 167

$$C = \frac{K\,.\,\theta\,.\,V}{1 + RK\,.\,\theta + LK\,.\,\theta^2} = \frac{K\theta}{(1 - LKn^2) + RK\theta}\,V$$

$$C = \frac{KnV_0}{\sqrt{(1 - LKn^2)^2 + R^2K^2n^2}} \sin\left(nt + \frac{\pi}{2} - \tan^{-1}\frac{RKn}{1 - LKn^2}\right).$$

The earnest student will take numbers and find out by much numerical trial what this means. If he were only to work this one example, he would discover that he now has a weapon to solve a problem in a few lines which some writers solve in a great many pages, using the most involved mathematical expressions, very troublesome, if not impossible, to follow in their physical meaning. Here the physical meaning of every step will soon become easy to understand.

† *Numerical Exercise.* Take $V_0 = 1414$ volts, $K = 1$ microfarad or 10^{-6}, $R = 100$ ohms and $n = 1000$, and we find the following effects produced by altering L. We give the following table and the curves in fig. 87:

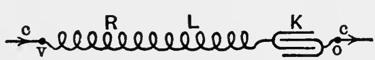

Fig. 86.

ABCD shows how the current increases slowly at first from *A* where $L = 0$ as L is increased, and then it increases more rapidly, reaching a maximum when $L = 1$ Henry and diminishing again exactly in the way in which it increased. *EFG* shows the lead which at $L = 1$ changes rather rapidly to a lag. The maximum current (when $LKn^2 = 1$) is the same as if we had no condenser and no self-induction, as if we had a mere non-inductive resistance R. It is interesting to note in the electric analogue of Art. 160 that this $LKn^2 = 1$ is the relation which would hold between L, K and n (neglecting the

small resistance term) if the condenser were sending surging currents through the circuit R, L, connecting its two coatings.

L, in Henries.	Effective current, in ampères.	Lead of current, in degrees.	L, in Henries.	Effective current, in ampères.	Lead of current, in degrees.
0	0·995	84·28	1·05	8·944	−26·57
0·1	1·110	83·67	1·1	7·071	−45·0
0·2	1·240	82·87	1·2	4·472	−63·43
0·3	1·414	81·87	1·3	3·162	−71·57
0·4	1·644	80·53	1·4	2·425	−75·97
0·5	1·961	78·67	1·5	1·961	−78·67
0·6	2·425	75·97	1·6	1·644	−80·53
0·7	3·162	71·57	1·7	1·414	−81·87
0·8	4·472	63·43	1·8	1·240	−82·87
0·9	7·071	45·0	1·9	1·110	−83·67
0·95	8·944	26·57	2·0	0·995	−84·28
0·975	9·701	14·03	2·5	0·665	−86·18
1·00	10·00	0	3·0	0·499	−87·13
1·025	9·701	−14·03			

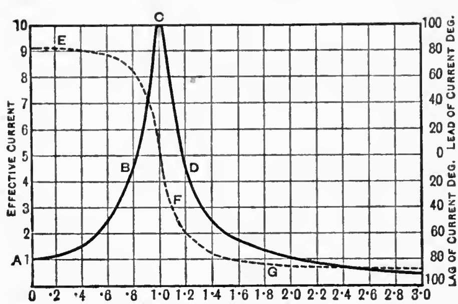

Fig. 87.

Experimenting with numbers as we have done in this example is much cheaper and much more conclusive in preliminary work on a new problem, than experimenting with alternators, coils and condensers.

172. Even if a transformer has its secondary open there is power being wasted in hysteresis and eddy currents, and the effect is not very different from what we should have if there was no such internal loss, but if there was a small load on. Assume, however, no load. **Find the effect of a condenser shunt in supplying the "Idle Current."**

The current to an unloaded transformer, consists of the fundamental term of the same frequency as the primary voltage, and other terms of three and five times the frequency, manufactured by the iron in a curious way. With these "other terms" the condenser has nothing to do; it cannot disguise them in any way; the total current always contains them. We shall not speak of them, as they may be imagined added on, and this saves trouble, for if the fundamental term only is considered we may imagine the permeability constant; that is, that the primary circuit of an unloaded transformer has simply a constant self-induction.

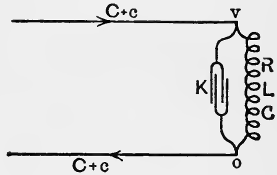

Fig. 88.

In fact between the ends of a coil (fig. 88) which has resistance R and self-induction L, place a condenser of capacity K. Let the voltage between the terminals, be $V = V_0 \sin nt$. Let C be the instantaneous current through the coil and let c be the current through the condenser, then $C + c$ is the current supplied to the system.

Now $$C = \frac{V_0 \sin nt}{R + L\theta},$$

and $$c = V_0 \sin nt \div \frac{1}{K\theta}, \text{ or } c = KnV_0 \cos nt,$$

$$C + c = \left(\frac{1}{R + L\theta} + K\theta\right) V_0 \sin nt$$

$$= \frac{1 + RK\theta + LK\theta^2}{R + L\theta} V = \frac{1 - LKn^2 + RK \, . \, \theta}{R + L\theta} V,$$

by our rule of Art. 167.

It is quite easy to write out by Art. 167 the full value of

$C+c$, but as we are not concerned now with the lag or lead, we shall only state the amplitude. It is evidently

$$V_0\sqrt{\frac{(1-LKn^2)^2+R^2K^2n^2}{R^2+L^2n^2}},$$

and the *effective* value of $C+c$ (what an ammeter would give as the measure of the current), is this divided by $\sqrt{2}$.

Observe that $C+c$ is least when

$$K=L/(R^2+L^2n^2).$$

(Note that if L is in Henries and $n=2\pi\times$ frequency (so that in practice $n=$ about 600), K is in farads. Now even a condenser of $\frac{1}{3}$ microfarad or $\frac{1}{3}\times 10^{-6}$ farad costs a number of pounds sterling. We have known an unpractical man to suggest the practical use of a condenser that would have cost millions of pounds sterling.)

When this is the case, the effective current $C+c$, is $R/\sqrt{R^2+L^2n^2}$ times the effective value of C.

The student ought to take a numerical case. Thus in an actual Hedgehog Transformer we have found $R=24$ ohms, $L=6{\cdot}23$ Henries $n=509$, corresponding to a frequency of about 81·1 per second. The effective voltage, or $V_0\div\sqrt{2}$ is 2400 volts. In fig. 89 we show the effective current calculated for various values of K. The current curve $ABCD$ is a hyperbola which is undistinguishable except just at the

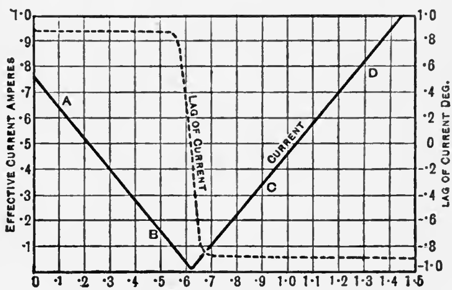

Fig. 89.

vertex, from two straight lines. The total current is a minimum when $K=L/(R^2+L^2n^2)$, in this case 0·618 micro-

farad; and the effect of the condenser has been to diminish the total current in the ratio of the resistance to the impedance. It is interesting on the curve to note how the great lag changes very suddenly into a great lead.

173. If currents are steady and if points A and B are connected by parallel resistances r_1, r_2, r_3, if V is the voltage between A and B, and if the three currents are c_1, c_2, c_3, and if the whole current is C; then

$$c_1 = \frac{V}{r_1}, \quad c_2 = \frac{V}{r_2}, \quad c_3 = \frac{V}{r_3},$$

$$C = V\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right).$$

In fact the three parallel conductors act like a conductance

$$\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right).$$

Also if C is known, then

$$c_1 = \frac{C}{\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right) r_1}.$$

Now let there be a self-induction l and a condenser of capacity k in each branch, and we have exactly the same instantaneous formulæ if, for any value of r, we insert

$$r + l\theta + \frac{1}{k\theta}.$$

The algebraic expressions are unwieldy, and hence numerical examples ought to be taken up by students.

174. Two circuits in parallel. They have resistances r_1 and r_2 and self-inductions l_1 and l_2. how does a total current C divide itself between them?

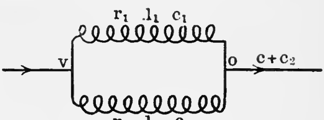

Fig. 90.

If the current were a continuous current, c_1 (fig. 90) in the branch r_1 would be

$$c_1 = \frac{r_2}{r_1 + r_2} C.$$

Hence it is now $c_1 = \dfrac{r_2 + l_2\theta}{r_1 + r_2 + (l_1 + l_2)\theta} C$.

If $C = C_0 \sin nt$, then by Art. 167

$$c_1 = C_0 \sqrt{\frac{r_2^2 + l_2^2 n^2}{(r_1 + r_2)^2 + (l_1 + l_2)^2 n^2}}$$
$$\sin\left(nt + \tan^{-1}\frac{l_2 n}{r_2} - \tan^{-1}\frac{(l_1 + l_2)n}{r_1 + r_2}\right).$$

In the last case suppose that for some instrumental purpose **we wish to use a branch part of C, but with a lead.** We arrange that $\tan^{-1}\dfrac{l_2 n}{r_2} - \tan^{-1}\dfrac{(l_1 + l_2)n}{r_1 + r_2}$ shall be equal to the required lead, and we use the current in the branch r_1 for our purpose.

175. Condenser annulling effects of self-induction. When the **voltage** between points A and B **follows any law whatever,** and we wish the current flowing into A and out at B to be exactly $\dfrac{V}{R}$, whatever V may be, and when we have already between A and B a coil of resistance R and self-induction L, show how to arrange a condenser shunt to effect our object.

Connect A and B by a circuit containing a resistance r, self-induction l and condenser of capacity K, as in fig. 91.

The total current is evidently

$$\frac{V}{R + L\theta} + \frac{V}{r + l\theta + \dfrac{1}{K\theta}};$$

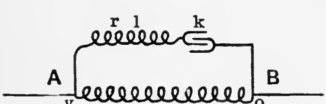

Fig. 91.

or, bringing all to a common denominator and arranging terms, it is

$$\frac{1 + \theta(rK + RK) + \theta^2(lK + LK)}{R + \theta(RrK + L) + \theta^2(RlK + LrK) + LlK\theta^3} V.$$

Observe that as V may be any function whatsoever of the time we cannot simplify this operator as we did those of

Art. 167. Now we wish the effect of the operation to be the same as $\frac{1}{R}V$. Equating and clearing of fractions we see that

$$R+\theta(RrK+R^2K)+\theta^2(RlK+RLK)$$

must be identical with

$$R+\theta(RrK+L)+\theta^2(RlK+LrK)+LlK\theta^3.$$

As V may be any function whatsoever of the time, the operations are not equivalent unless $LlK=0$; that is, $l=0$; so there must be no self-induction in the condenser circuit,

$$RrK+R^2K=RrK+L;\ \text{that is, } K=\frac{L}{R^2};$$

$$RlK+RLK=RlK+LrK;\ \text{that is, } R=r;$$

so the resistance in the condenser circuit must be equal to that in the other.

In fact we must shunt the circuit $\mathbf{R+L\theta}$ by a condenser circuit $\mathbf{R+\frac{1}{K\theta}}$ where $\mathbf{K=\frac{L}{R^2}}$.

176. If in the last case $V=V_0\sin nt$, the operator may be simplified into

$$\frac{1-K(l+L)n^2+K(r+R)\theta}{R-K(Rl+rL)n^2+\theta\{RrK+L-LlKn^2\}};$$

and if $$\frac{R-K(Rl+rL)n^2}{1-K(l+L)n^2}=\frac{RrK+L-LlKn^2}{K(r+R)},$$

then although the adjustment alters when frequency alters, we have for a fixed value of n the current flowing in at A and out at B proportional to V and without any lag. If $R=r$ the current is equal to $\frac{V}{R}$.

177. To explain why the **effective voltage is sometimes less** between the mains at a place D, fig. 92, **than at a place B further away from the generator.** This is usually due to a distributed capacity (about $\frac{1}{3}$ microfarad per mile is usual) in the mains. We may consider a distributed capacity later; at present assume one condenser of

capacity K between the mains at B. Let the non-inductive resistance, say of lamps, between the mains at B be r. Let the resistance and self-induction of the mains between D and B be R and L. Let v be the voltage at B, and C the current from D to B.

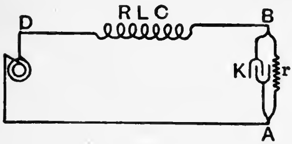

Fig. 92.

The current into the condenser is $v \div \frac{1}{K\theta}$ or $K\theta v$.

The current through r is $\frac{v}{r}$, so that

$$C = \left(K\theta + \frac{1}{r}\right) v \text{......................(1).}$$

The *drop* of voltage between D and B is

$$(R + L\theta)\, C \text{ or } (R + L\theta)\left(K\theta + \frac{1}{r}\right) v,$$

or

$$\left\{\frac{R}{r} + \left(RK + \frac{L}{r}\right)\theta + LK\theta^2\right\} v.$$

Now if $v = v_0 \sin nt$, the *drop* is

$$\left\{\left(\frac{R}{r} - LKn^2\right) + \left(RK + \frac{L}{r}\right)\theta\right\} v.$$

The voltage at D is the *drop* plus v, or

$$\left\{\left(1 + \frac{R}{r} - LKn^2\right) + \left(RK + \frac{L}{r}\right)\theta\right\} v;$$

also by Art. 167, $\dfrac{\text{square of effective voltage at } D}{\text{square of effective voltage at } B}$

$$= \left(1 + \frac{R}{r} - LKn^2\right)^2 + \left(RK + \frac{L}{r}\right)^2 n^2,$$

and there are values of the constants for which this is less than 1. As a numerical example take

$$r = 10,\ R = \cdot 1,\ K = 1 \times 10^{-6},\ n = 1000,$$

and let L change from 0 to ·05, ·01, ·02, ·03 &c.

The student will find no difficulty in considering this problem when $r + l\theta$ is used instead of r in (1); that is, when not merely lamps are being fed beyond B, but also coils having self-induction.

Most general case of Two Coils.

178. Let there be a coil, fig. 93, with electromotive force E, resistance R, self-induction L, capacity K; and another with e, r, l, k. Let the mutual induction be m.

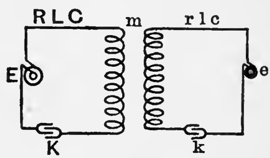

Fig. 93.

Using then R for $R + L\theta + \dfrac{1}{K\theta}$,

and $\qquad r$ for $r + l\theta + \dfrac{1}{k\theta}$,

the equations are

$$\left.\begin{aligned} E &= RC + m\theta c, \\ e &= m\theta C + rc \end{aligned}\right\} \text{.......................(1).}$$

Notice how important it is for a student not to trouble himself about the signs of C and c &c. until he obtains his answers.

From these we find

$$c = \frac{Re - m\theta E}{Rr - m^2\theta^2} \text{.......................(2),}$$

$$C = \frac{rE - m\theta e}{Rr - m^2\theta^2} \text{.......................(3).}$$

We can now substitute for R, r, E and e their values and obtain the currents.

Observe that E may be a voltage established at the terminals of part of a circuit, and then R is only between these terminals.

The following exercises are examples of this general case.

There are a great many other examples in which mutual induction comes in.

179. Let two circuits (fig. 94), **with self-inductions,**

be in parallel, with mutual induction m between them.

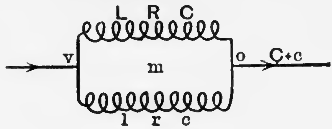

Fig. 94.

(1) At their terminals let $v = v_0 \sin nt$; (2) of last exercise becomes $c = \dfrac{R - m\theta}{Rr - m^2\theta^2} v$. Or, changing R into $R + L\theta$ and r into $r + l\theta$,

$$c = \frac{R + (L - m)\theta}{\{Rr - (Ll - m^2) n^2\} + (Lr + lR)\theta} v.$$

(2) How does a current $A \sin nt$ divide itself between two such circuits? Since $\dfrac{c}{C} = \dfrac{R - m\theta}{r - m\theta}$ we can find at once $\dfrac{c}{C + c}$ and $\dfrac{C}{C + c}$. Answer: $c = \dfrac{R + (L - m)\theta}{(R + r) + (L + l - 2m)\theta}$ operating on $A \sin nt$.

We think it is hardly necessary to work such examples out more fully for students, as, to complete the answers they have the rule in Art. 167.

180. In the above example, **imagine each of the circuits to have also a mutual induction with the compound circuit.** We shall use new letters as shown in fig. 95.

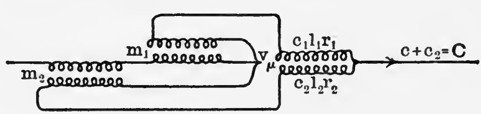

Fig. 95.

If v is the potential difference between the ends of the two circuits which are in parallel. Using r, μ and m to stand for $r + l\theta$, $\mu\theta$ and $m\theta$,

$$v = r_1 c_1 + \mu c_2 + m_1 (c_1 + c_2).$$

Hence the equations are

$$v = (r_1 + m_1)\, c_1 + (\mu + m_1)\, c_2,$$

$$v = (\mu + m_2)\, c_1 + (r_2 + m_2)\, c_2,$$

$$c_1 = \frac{(r_2 + m_2) - (\mu + m_1)}{(r_1 + m_1)(r_2 + m_2) - (\mu + m_1)(\mu + m_2)}\, v \quad \ldots(1),$$

with a similar expression for c_2.

Also a total current C divides itself in the following way

$$c_1 = \frac{(r_2 + m_2) - (\mu + m_1)}{r_1 + r_2 - 2\mu}\, C \ldots\ldots\ldots\ldots\ldots(2).$$

If we write these out in full, we have exceedingly pretty problems to study, and our study might perhaps be helped by taking numerical values for some of the quantities. If we care to introduce condensers, we need only write $r + l\theta + \dfrac{1}{k\theta}$ with proper affixes, instead of each r; μ becomes $\mu\theta$ and m becomes $m\theta$.

To what extent may we make some of the m's negative? I have not considered this fully, but some student ought to try various values and afterwards verify his results with actual coils. Taking (2) without condensers

$$c_1 = \frac{r_2 + \theta\,(l_2 + m_2 - \mu - m_1)}{r_1 + r_2 + \theta\,(l_1 + l_2 - 2\mu)}\, C.$$

181. Rotating Field. Current passes through a coil wound on a non-conducting bobbin; the same current passes through a coil wound on a conducting bobbin. The coils are at right angles and have no mutual induction; find the nature of the fields which are at right angles at the centre of the two bobbins. Let the numbers of turns be n_1 and n_2. Instead of a conducting bobbin imagine a coil closed on itself of resistance r_3, and n_3 turns and current c. For simplicity, suppose all three mean radii the same, and the coils n_2 and n_3 well intermingled. One field F_1 is proportional to n_1C per square centimetre, call it n_1C. The other F_2 is proportional, or let us say equal to, $n_2C + n_3c$ per square cm. Take the total induction I through each of the coils as

proportional to the intensity of field at its centre, say b times. Then for the third coil, we have

$$0 = r_3 c + n_3\theta I \text{ or } = r_3 c + bn_3\theta(n_2 C + n_3 c),$$

so that
$$-c = \frac{bn_3 n_2 \theta C}{r_3 + bn_3^2\theta},$$

and hence
$$F_2 = n_2 C - \frac{bn_3^2 n_2 \theta C}{r_3 + bn_3^2\theta} = \frac{n_2 r_3 C}{r_3 + bn_3^2\theta}.$$

If then $C = C_0 \sin qt$,

$$F_1 = n_1 C_0 \sin qt,$$

$$F_2 = \frac{n_2 C_0 \sin qt}{1 + b\dfrac{n_3^2}{r_3}\theta} = \frac{n_2 C_0}{\sqrt{1 + b^2\dfrac{n_3^4}{r_3^2}q^2}} \sin\left(qt - \tan^{-1}\frac{bn_3^2}{r_3}q\right):$$

Art. 126, shows the nature of the rotating field. We can assure the student that he may obtain an excellent rotating field in this way.

It is evident that bn_3^2 really means the self-induction of the third coil, and $\dfrac{bn_3^2}{r_3}$ means its time constant. A coil of one turn,—that is, a conducting bobbin, will have a greater time constant than any coil of more than one turn wound in the same volume. It is evident that if the bobbin is made large enough in dimensions, we can for a given frequency have an almost uniform and uniformly rotating field by making

$$n_2 \div b\frac{n_3^2}{r_3}q = n_1.$$

This is one of a great number of examples which we might give to illustrate the usefulness of our sign of operation θ.

182. In Art. 178 let $E = V$ the primary voltage of **a transformer**, the primary circuit having internal resistance R and self-induction L; let the secondary have no independent E.M.F. in it; let its internal resistance be r_1 and self-induction l and let it have an outside non-inductive resistance ρ, of lamps. Let the voltage at the secondary terminals be $v = c\rho$.

Then in (1), (2) and (3) Art. 178, let $E=V$, $e=0$; instead of R use $R+L\theta$. Instead of r use $r+l\theta$ which is really $r_1+\rho+l\theta$,

$$c=\frac{-m\theta V}{Rr+(Rl+rL)\,\theta+(Ll-m^2)\,\theta^2}\;\text{.........}(1),$$

$$C=\frac{(r+l\theta)\,V}{Rr+(Rl+rL)\,\theta+(Ll-m^2)\,\theta^2}\;\text{.........}(2).$$

Note that the second equation of (1) Art. 178 is

$$0=m\theta C+(r+l\theta)\,c\;\text{..............}(2)^*.$$

I. From (2)*, if $\mathbf{C=C_0\epsilon^{at}}$, $c=\dfrac{-maC_0\epsilon^{at}}{r+la}$,

$$\frac{-c}{C}=\frac{ma}{r+la}=\frac{m}{l}\left(\frac{1}{1+\dfrac{r}{la}}\right).$$

If r is small compared with la

$$\frac{-c}{C}=\frac{m}{l}.$$

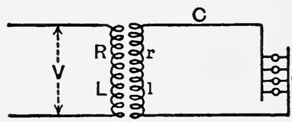

Fig. 96.

II. If $\mathbf{C=C_0\sin qt}$, again using (2)*

$$-c=\frac{mq}{\sqrt{r^2+l^2q^2}}C_0\sin\left(qt+\frac{\pi}{2}-\tan^{-1}\frac{lq}{r}\right).$$

Hence $$\frac{\text{effective } c}{\text{effective } C}=\frac{m}{l}\,\frac{1}{\sqrt{1+\dfrac{r^2}{l^2q^2}}}.$$

Except when the load on the secondary is less than it ever is usually in practice, r is insignificant compared with lq (a practical example ought to be tried to test this) and we may take

$$C=C_0\sin qt,$$

$$c=C_0\frac{m}{l}\sin(qt-\pi),$$

or $$\mathbf{\frac{-c}{C}=\frac{m}{l}}\;\text{........................}(3).$$

It may become important in some application to remember that the ratio of the instantaneous values of $-c$ and C is that of

$$r \sin qt + lq \cos qt \text{ to } mq \cos qt,$$

and this sometimes is ∞. †

Returning to (1). Let $Ll = m^2$ (this is the condition called **no magnetic leakage**) and let Rr be negligible. In any practical case, Rr is found to be negligible even when r is so great as to be several times the resistance of only one lamp.†

Then $$-c = \frac{mV}{Rl + rL} \quad \text{.......................(4)},$$

so that $-c$ is a faithful copy of V as a function of the time. C is so also.

If N and n are the numbers of windings of the two coils on the same iron,

$$m : L : l = Nn : N^2 : n^2 \quad \text{.................(5)},$$

so that $$-c = \frac{\frac{n}{N} V}{r + R \frac{n^2}{N^2}} \quad \text{...................... (6)};$$

that is, the secondary current is the same as if the transformed voltage $\left(-\frac{n}{N} V\right)$ acted in the secondary circuit, but as if an extra resistance were introduced which I call the transformed primary resistance $\left(R \frac{n^2}{N^2}\right)$.

If the volumes of the two coils were equal, and if the volumes of their insulations were equal, $R \frac{n^2}{N^2}$ would be equal to r_1 the internal resistance of the secondary. Assume it so and then

$$-c = \frac{\frac{n}{N} V}{2r_1 + \rho} \quad \text{........................(7)};$$

also ρc or
$$v = -\frac{\frac{n}{N}V}{1+\frac{2r_1}{\rho}} \text{.......................(8).}$$

As r_1 is usually small compared with ρ,
$$-v = \frac{n}{N}V\left(1-\frac{2r_1}{\rho}\right),$$

and $\frac{2r_1}{\rho}$ is called the drop in the secondary voltage due to load.

As $\frac{v^2}{\rho} = P$, the power given to lamps; $\frac{1}{\rho} = \frac{P}{v^2}$ and the fractional *drop* is $\frac{2r_1}{v^2}P$ and is proportional to the Power, or to the number of lamps which are in circuit.

183. The above results may be obtained in another way.
Let I be the induction, and let it be the same in both coils. Here again we assume no magnetic leakage,
$$V = RC + N\theta I \text{.......................(1),}$$
$$0 = rc + n\theta I \text{.......................(2).}$$
Multiplying each equation by its N or n and dividing by its R or r and adding
$$\frac{NV}{R} = A + \left(\frac{N^2}{R}+\frac{n^2}{r}\right)\theta I \text{.................(3),}$$
where $A = NC + nc$, and is called the current turns.

Now when we know the nature of the magnetic circuit, that is, the nature of the iron and its section, α square centimetres, and the average length λ centimetres of the magnetic circuit, we know the relationship between A and I. I have gone carefully into this matter and find that whatever be the nature of the periodic law for A, so long as the frequency and sizes of iron &c. are what they usually are in practice, the term A is utterly insignificant in (3). Rejecting it we find
$$I = \frac{\theta^{-1}V}{N\left(1+\frac{n^2}{N^2}\frac{R}{r}\right)} = \frac{1}{N}\left(1-\frac{n^2}{N^2}\frac{R}{r}\right)\theta^{-1}V \text{ very nearly ...(4).}$$

Thus, in a certain 1500-watt transformer, $R = 27$ ohms, $N = 460$ turns, internal part of $r = \cdot 067$ ohms, $n = 24$ turns, effective V is 2000 volts or $V = 2828 \sin qt$ where $q = 600$ say, $\alpha = 360$, $\lambda = 31$. When there is no load $r = \infty$; on full load $r =$ nearly 7 ohms.

We have called $R\dfrac{n^2}{N^2}$ the transformed resistance of the primary. It is in this case $27\left(\dfrac{24}{460}\right)^2$ or $\cdot 073$ ohms.

If the primary and secondary volumes of copper† had been equal, no doubt this would have been more nearly identical with $\cdot 067$, the internal resistance of the secondary.

$\dfrac{n^2R}{N^2r}$ or $\dfrac{\cdot 073}{r}$ is the fractional drop in I from what it is at no load. When at full load $r = 7$ ohms the fractional drop is greatest, and it is only 1 per cent. in this case. Because of its smallness we took a fractional increase of the denominator as the same fractional diminution of the numerator of (4).

Consider I at its greatest, that is, at no load; $\dfrac{1}{\theta}V$ is the integral of V or $-\dfrac{2828}{600}\cos 600t$. So that **the amplitude of I** is $\dfrac{2828}{600 \times 460}$.

Multiply this, the maximum value of I in Webers, by 10^8 to obtain C. G. S. units, and divide by $\alpha = 360$, and we find 2856 C. G. S. units of induction per sq. cm. in the iron, as the maximum in this transformer every cycle.

θI being $\dfrac{1}{N}V \Big/ \left(1 + \dfrac{n^2}{N^2}\dfrac{R}{r}\right)$, we have from (2) the same value of $-rc$ that we had before in (6) of Art. 183.

184. Returning to (7) of Art. 182. Let us suppose that **there is magnetic leakage** and that r_1 is really $r_1 + l'\theta$. If one really goes into the matter it will be seen that this is what we mean by magnetic leakage. Then we must divide by

$$\rho + 2r_1 + 2l'\theta,$$

instead of $\rho + 2r_1$. In fact our old answer must be divided by

$$1 + \frac{2l'}{\rho + 2r_1}\theta,$$

or neglecting $2r_1$ as not very important in this connection; our old answer must be divided by $1+\frac{2l'}{\rho}\theta$. This means that the old amplitude of v must be divided by

$$\sqrt{1+\frac{4l'^2q^2}{\rho^2}} \text{ or } 1+\frac{2l'^2q^2}{\rho^2} \text{ nearly,}$$

if the leakage is small, and there is a lag produced of the amount $\tan^{-1}\frac{2l'q}{\rho}$. We must remember that q is $2\pi f$ if f is the frequency. We saw that P, the power given to the lamps, is inversely proportional to ρ, so we see that the **fractional drop** due to **mere resistances** is $\frac{2r_1P}{v^2}$, the fractional drop due to **magnetic leakage** is $\frac{1}{2}a^2f^2P^2$, and **the lag** due to magnetic leakage is an angle of afP radians where a is a constant which depends upon the amount of leakage, and f is the frequency.

185. Only one thing need now be commented upon in regard to Transformers. If V is known, it has only to be integrated and divided by N to get I. Multiply by 10^8 and divide by the cross-section of the iron in square centimetres, and we know how β, the induction per sq. cm. in the iron, alters with the time. The experimentally obtained β, H curve for the iron enables us to find for every value of β the corresponding value of H, and H multiplied by the length of the magnetic circuit in the iron gives the gaussage, or $\frac{4\pi}{10}\times$ the ampère turns A. Hence the law of variation of A is known, and if there is no secondary current, we have **the law of the primary current** in an unloaded transformer or choking coil. This last statement is, however, inaccurate, as one never has a truly unloaded transformer, even when what is usually called the secondary, has an infinite resistance.

186. Sir W. Grove's Problem; the effect of a condenser in the primary of **an induction coil** when using alternating currents.

ADB, fig. 97, is the primary with electromotive force $E=E_0\sin nt$, resistance R and self-induction L. BA is a

condenser of capacity K, and r is a non-inductive resistance in parallel with the condenser. C the current in the primary, has an amplitude C_0, say.

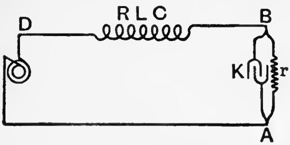

Fig. 97.

The condenser has the resistance $\frac{1}{K\theta}$.

It is quite easy to write out the value of C_0 when r and K have any finite values*.

But for our problem we suppose $r=0$ or else $r=\infty$. When $r=0$, the resistance is $R+L\theta$ and the current is $E/(R+L\theta)$,

$$C_0^2 = \frac{E_0^2}{R^2+L^2n^2} \dots\dots\dots\dots (1).$$

When $r=\infty$, the resistance is

$$R+L\theta+\frac{1}{K\theta} \text{ or } \frac{1+RK\theta+LK\theta^2}{K\theta};$$

or

$$\frac{(1-LKn^2)+RK\theta}{K\theta}, \text{ by Art. 167,}$$

and

$$C_0^2 = \frac{E_0^2K^2n^2}{(1-LKn^2)^2+R^2K^2n^2} = \frac{E_0^2}{R^2+\left(\frac{1}{Kn}-Ln\right)^2} \dots (2).$$

Now (2) is greater than (1) if $2KLn^2$ is greater than 1, so that the primary current is increased by a condenser of capacity greater than $\frac{1}{2Ln^2}$. Again, there is a maximum current if $K=\frac{1}{Ln^2}$; in this case the condenser completely destroys the self-induction of the primary.

* When both r and K have finite values, the parallel resistances between B and A, together form a resistance $r/(1+rK\theta)$, and the whole resistance of the circuit for C is $R+L\theta+\frac{r}{1+rK\theta}$ so that

$$C = \frac{(1+rK\theta)\,E_0 \sin nt}{(R+r-LrKn^2)+(RrK+L)\,\theta},$$

$$C_0^2 = \frac{(1+r^2K^2n^2)\,E_0^2}{(R+r-LrKn^2)^2+(RrK+L)^2n^2},$$

and the lag of C is easily written.

187. Alternators in series. Let their E.M.F. be e_1 and e_2 and let C be the current through both. The powers exerted are e_1C and e_2C. Now if

$e_1=E\sin(nt+\alpha)$ and $e_2=E\sin(nt-\alpha)$,† $e_1+e_2=2E\cos\alpha\,.\,\sin nt$.

If l is the self-induction of each machine, r its internal resistance, and if $2R$ is the outside resistance and if P_1 and P_2 are the average powers developed in the two machines,

$$C=\frac{2E\cos\alpha\,.\,\sin nt}{2l\theta+2r+2R}=\frac{E\cos\alpha}{\sqrt{(R+r)^2+l^2n^2}}\sin\left(nt-\tan^{-1}\frac{ln}{R+r}\right)$$

$$=M\cos\alpha\sin(nt-\epsilon)\text{ say,}$$

$$P_1=\tfrac{1}{2}ME\cos\alpha\,.\cos(\alpha+\epsilon),$$

$$P_2=\tfrac{1}{2}ME\cos\alpha\,.\cos(\alpha-\epsilon).$$

Hence P_2 is greater than P_1, and machine 2 is retarded whilst machine 1 is accelerated; hence α increases until $\alpha=\frac{\pi}{2}$, and when this is the case, $\cos\alpha=0$, so that $P_1=0$, $P_2=0$ and the machines neutralize each other, producing no current in the circuit. **Alternators cannot therefore be used in series** unless their shafts are fastened together.

188. As we very often have to deal with circuits in parallel we give the following general formula; if the electromotive forces e_1, e_2 and e_3, fig. 98, are constant,

$$v=e_1-c_1r_1=e_2-c_2r_2=e_3-c_3r_3\ \dots\dots\dots\dots(1),$$

and

$$c_1+c_2+c_3=0\dots\dots\dots\dots\dots\dots\dots\dots(2).$$

Given the values of e_1, e_2, e_3 and r_1, r_2, r_3 we easily find the currents, because

$$v=\left(\frac{e_1}{r_1}+\frac{e_2}{r_2}+\frac{e_3}{r_3}\right)\div\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)\dots\dots\dots\dots(3).$$

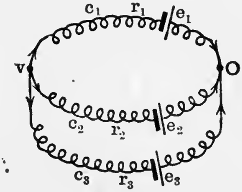

Fig. 98.

Now if the e's are not constant, we must use $r_1 + l_1\theta$, &c., instead of mere resistances.

189. Alternators in Parallel. Let two alternators, each of resistance r and self-induction l, and with electromotive forces, $e_1 = E\sin(nt+\alpha)$, and $e_2 = E\sin(nt-\alpha)$, be coupled up in parallel to a non-inductive circuit of resistance R. What average electrical power will each of them create, and will they tend to synchronism? If e_1 and e_2 were constant or if l were 0, then $v = e_1 - c_1 r = e_2 - c_2 r = (c_1 + c_2) R$.

And hence

$$c_1 = \frac{R}{2rR + r^2}\left\{e_1\left(1 + \frac{r}{R}\right) - e_2\right\},$$

$$c_2 = \frac{R}{2rR + r^2}\left\{e_2\left(1 + \frac{r}{R}\right) - e_1\right\}.$$

Now alter r to $r + l\theta$, because the e's are alternating. The student will see that we may write

$$e_2 = e_1(a - b\theta),$$

$$e_1 = e_2(a + b\theta),$$

where $a^2 + b^2n^2 = 1$, $a = \cos 2\alpha$, $bn = \sin 2\alpha$. Then

$$c_1 = \frac{\left(1 + \frac{r}{R} - a\right) + \theta\left(\frac{l}{R} + b\right)}{\left(2r + \frac{r^2}{R} - \frac{l^2n^2}{R}\right) + \theta 2l\left(1 + \frac{r}{R}\right)} e_1 \quad \ldots\ldots(1),$$

with a similar expression for c_2 in terms of e_2 except that b is made negative. If we write out (1) by the rule of Art. 167, there is some such simplification as this:—

Let $\tan\phi = \dfrac{2ln(R+r)}{2Rr + r^2 - l^2n^2}$ and $\tan\psi_1 = \dfrac{(l + bR)n}{R + r - aR}$

$$\tan\psi_2 = \frac{(l - bR)n}{R + r - aR}.$$

Then $$c_1 = M\sin(nt + \alpha - \phi + \psi_1),$$

$$c_2 = M\sin(nt - \alpha - \phi + \psi_2),$$

the angles ϕ, ψ_1, ψ_2 being all supposed to be between 0 and $\pm 90°$.

The average powers are

$$P_1 = ME\cos(\phi - \psi_1),$$
$$P_2 = ME\cos(\phi - \psi_2),$$

where $$M^2 = \frac{\left(1+\frac{r}{R}\right)^2 - 2\left(1+\frac{r}{R}\right)\cos 2\alpha + 1 + \frac{l^2n^2}{R^2}}{\left(2r + \frac{r^2}{R} - \frac{l^2n^2}{R}\right)^2 + 4l^2n^2\left(1+\frac{r}{R}\right)^2} E^2,$$

$$\frac{P_1}{P_2} = \frac{\cos(\phi - \psi_1)}{\cos(\phi - \psi_2)}.$$

If $R = \infty$ we see that

$$\tan\phi = \frac{ln}{r}, \ \tan\psi_1 = \frac{\sin\alpha}{1 - \cos\alpha} = -\tan\psi_2.$$

Hence $$\frac{P_1}{P_2} = \frac{\cos(\phi - \psi_1)}{\cos(\phi + \psi_1)}.$$

In this case it is obvious that P_1 is greater than P_2. The author has not examined the general expression for $\frac{P_1}{P_2}$ with great care, himself, but men who have studied it say that it shows P_1 to be always greater than P_2. Students would do well to take values for r, l, R and α and try for themselves. If P_1 is always greater, it means that the leading alternator has more work to do, and it will tend to go slower, and the lagging one tends to go more quickly, so that there is a tendency to synchronism and **hence alternators will work in Parallel.**

190. Struts. Consider a strut perfectly prismatic, of homogeneous material, its own weight neglected, the resultant force F at each end passing through the centre of each end. Let ACB, fig. 99, show the centre line of the bent strut. Let $PQ = y$ be the deflection at P where $OQ = x$. Let $OA = OB = l$. y is supposed everywhere small in comparison with the length $2l$ of the strut.

Fy is the bending moment at P, and $\frac{Fy}{EI}$ is the curvature there, if E is Young's modulus for the material and I is

the least moment of inertia of the cross section everywhere, about a line through the centre of area of the section. Then as in Art. 60 the curvature being $-\frac{d^2y}{dx^2}$* we have

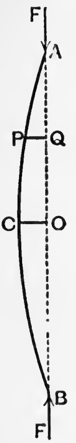

Fig. 99.

$$\frac{Fy}{EI} = -\frac{d^2y}{dx^2} \text{.....................(1).}$$

Now if the student tries he will find that, as in the many cases where we have had and again shall have this equation, (see Art. 119)

$$y = a \cos x\sqrt{\frac{F}{EI}} \text{ .... (2)}$$

satisfies (1) whatever value a may have. When $x=0$ we see that $y=a$, so that the meaning of a is known to us; it is the deflection of the strut in the middle. The student is instructed to follow carefully the next step in our argument.

When $x=l$, $y=0$. Hence

$$a \cos l\sqrt{\frac{F}{EI}} = 0 \text{ ....(3).}$$

---

* Notice that when we choose to call $\frac{d^2y}{dx^2}$ the curvature of a curve, if the expression to which we put it equal is essentially positive, we must give such a sign to $\frac{d^2y}{dx^2}$ as will make it also positive. Now if the slope of the curve of fig. 99 be studied as we studied the curve of fig. 6, we shall find that $\frac{d^2y}{dx^2}$ is negative from $x=0$ to $x=OA$, and as y is positive so that $\frac{Fy}{EI}$ is positive, we must use $-\frac{d^2y}{dx^2}$ on the right-hand side.

It will be found that the *complete* (see Arts. 154 and 159) solution of any such equation as (1) which may be written

$$\frac{d^2y}{dx^2} + n^2y = 0$$

is

$$y = A \cos nx + B \sin nx$$

where A and B are arbitrary constants. A and B are chosen to suit the particular problem which is being solved. In the present case it is evident that, as $y=0$ when $x=l$ and also when $x=-l$,

$$0 = A \cos nl + B \sin nl,$$

$$0 = A \cos nl - B \sin nl, \quad \text{so that } B \text{ is } 0.$$

Now how can this be true? Either $a=0$, or the cosine is 0. Hence, *if bending occurs,* so that a has some value, *the cosine must be* 0. Now if the cosine of an angle is 0 the angle must be $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ or $\frac{5\pi}{2}$, &c. It is easy to see why we confine our attention to $\frac{\pi}{2}$ *.

Hence **the condition that bending occurs** is

$$l\sqrt{\frac{F}{EI}}=\frac{\pi}{2}, \text{ or } \mathbf{F}=\frac{\mathbf{EI}\pi^2}{\mathbf{4l^2}} \quad\text{..............(4)}$$

is the load which will produce bending. This is called Euler's law of strength. The load given by (4) will produce either very little or very much bending equally well. It is very easy to extend the theory to struts fixed at both ends or fixed at one end and hinged at the other.

For equilibrium under exceedingly great bending, the equation (1) is not correct, as $\frac{d^2y}{dx^2}$ is not equal to the curvature when the curvature is great, but for all engineering purposes it may be taken as correct.

191. We may take it that F given by (4), is the load which will break a strut if it breaks by bending. If f is the compressive stress which will produce rupture and A is the area of cross section, the load fA will break the strut by direct crushing, and we must take the smaller of the two answers. In fact we see that fA is to be taken for short struts or for struts which are artificially† protected from bending, and (4) is to be taken for long struts. Now, even when great care is taken, we find that struts are neither quite straight nor homogeneous, nor is it easy to load them in the specified manner. Consequently when loaded, they deflect with **even small loads,** and they break with loads less than either fA or that given by (4).

---

* This gives the least value of W. The meaning of the other cases is that y is assumed to be 0 one or more times between $x=0$ and $x=l$, so that the strut has points of inflexion.

† This casual remark contains the whole theory of struts such as are used in the Forth Bridge.

Curiously enough, however, when struts of the same section but of different lengths are tested, their breaking loads follow, with a rough approximation to accuracy, some rule as to length. Let us assume that as $F = fA$ for short struts, and what is given in (4) for long struts, then the formula

$$F = \frac{fA}{1 + \frac{fA4l^2}{EI\pi^2}} \quad \text{.....................(5)}$$

may be taken to be true for struts of all lengths, because it is **true both for short and for long ones**. For if l is great we may neglect 1 in the denominator, and our (5) is really (4); again, when l is small, we may regard the denominator as only 1 and so we have $W = fA$. We get in this way an empirical formula which is found to be fairly right for all struts. To put it in its usual form, let $I = Ak^2$, k being the least radius of gyration of the section about a line through its centre of gravity, then

$$F = \frac{fA}{1 + a\frac{l^2}{k^2}} \quad \text{.......................(6)},$$

where a is $4f/E\pi^2$, or rather f and a are numbers best determined from actual experiments on struts.

If F does not act truly at the centre of each end, but at the distance h from it, our end condition is that $y = h$ when $x = l$. This will be found to explain why struts not perfectly truly loaded, break with a load less than what is given in (4). Students who wish to pursue the subject are referred to pages 464 and 513 of the *Engineer* for 1886, where initial want of straightness of struts is also taken account of.

192. Struts with Lateral Loads. We had better confine our attention to a strut with hinged ends. If the lateral loads are such that by themselves and the necessary lateral supporting forces, they produce a bending moment which we shall call $\phi(x)$, then (1) Art. 190 becomes

$$Fy + \phi(x) = -EI\frac{d^2y}{dx^2}.$$

Thus let a strut be uniformly loaded laterally, as by centrifugal force or its own weight, and then $\phi(x) = \frac{1}{2}w'(l-x)^2$ if w' is the lateral load per unit length.

We find it slightly more convenient to take $\phi(x)=\frac{1}{4}Wl\cos\frac{\pi}{2l}x$ where W is the total lateral load; this is not a very different law. Hence

$$\frac{d^2y}{dx^2}+\frac{F}{EI}y+\frac{1}{4}\frac{Wl}{EI}\cos\frac{\pi}{2l}x=0 \text{ .... (1).}$$

We find here that

$$y=\frac{\frac{1}{4}Wl}{EI\frac{\pi^2}{4l^2}-F}\cos\frac{\pi}{2l}x \text{ .... (2).}$$

Observe that when $F=0$ this gives the shape of the beam.

The deflexion in the middle is

$$y_1=\frac{\frac{1}{4}Wl}{EI\frac{\pi^2}{4l^2}-F} \text{ ....(3),}$$

and the greatest bending moment μ is

$$\mu=Fy_1+\tfrac{1}{4}Wl, \text{ or}$$

$$\mu=\tfrac{1}{4}Wl\,.\,\frac{EI\pi^2}{4l^2}\Big/\left(\frac{EI\pi^2}{4l^2}-F\right) \text{.....................(4).}$$

If $W=0$ and if μ has any value whatever, the denominator of (4) must be 0. Putting it equal to 0, we have Euler's law for the strength of struts which are so long that they bend before breaking. If Euler's value of F be called U, or $U=EI\pi^2/4l^2$, (4) becomes

$$\mu=\tfrac{1}{4}Wl\frac{U}{U-F} \text{ ....(5).}$$

If z_c is the greatest distance of a point in the section from the neutral line on the compressive side, or if $I\div z_c=Z$, the least strength modulus of the section, and A is the area of cross section, and if f is the maximum compressive stress to which any part of the strut is subjected,

$$\frac{\mu}{Z}+\frac{F}{A}=f.$$

Using this expression, if β stands for $\frac{U}{A}$ (that is Euler's Breaking load per square inch of section), and if w stands for $\frac{F}{A}$ (the true breaking load per inch of section), then

$$\left(\mathbf{1}-\frac{\mathbf{w}}{\mathbf{f}}\right)\left(\mathbf{1}-\frac{\mathbf{w}}{\boldsymbol{\beta}}\right)=\frac{\mathbf{Wl}}{\mathbf{4fZ}} \text{ .... (6).}$$

This formula is not difficult to remember. From it w may be found.

Example. Every point in an iron or steel **coupling rod,** of length $2b$ inches, moves about a radius of r inches. Its section is rectangular,

d inches in the plane of the motion and b at right angles to this. We may take $W=lbdrn^2 \div 62940$, in pounds, where n=number of revolutions per minute. Take it as a strut hinged at both ends, for both directions in which it may break.

1st. For bending in the direction in which there is no centrifugal force where I is $\frac{db^3}{12}$,

Euler's rule gives $\qquad \frac{Edb^3\pi^2}{48l^2}$ ....(7).

Now we shall take this as the endlong load which will cause the strut to break in the other way of bending also, so as to have it equally ready to break both ways.

2nd. Bending in the direction in which bending is helped by centrifugal force. Our w of (6) is the above quantity of (7) divided by bd, or taking

$$E=3\times 10^7,$$

$$w=6{\cdot}17\times\frac{b^2}{l^2}\times 10^6.$$

Taking the proof stress f for the steel used, as 20000 lb. per sq. inch (remember to keep f low, because of reversals of stress), and recollecting the fact that I in this other direction is $\frac{bd^3}{12}$, we have (6) becoming

$$8{\cdot}4\times 10^8\left(1-308\frac{b^2}{l^2}\right)\left(1-\frac{b^2}{d^2}\right)=n^2l^2r\div d \text{ .... (8).}$$

Thus for example, if $b=1$, $l=30$, $r=12$, the following depths d inches, are right for the following speeds. It is well to assume d and calculate n from (8).

d	1	1·5	2	2·5	3	4	6
n	0	205	277	327	368	437	545

Exercise. A round bar of steel, 1 inch in diameter, 8 feet long, or $l=48$ inches. Take $F=1500$ lb. Show that an endlong load only sufficient of itself to produce a stress of 1910 lb. per sq. in., and a bending moment which by itself would only produce a stress of 816 lb. per sq. inch; if both act together, produce a stress of 23190 lb. per sq. inch.

For other interesting examples the student is referred to *The Philosophical Magazine* for March, 1892. †

CHAPTER III.

ACADEMIC EXERCISES.

193. In Chapter I. we dealt only with the differentiation and integration of x^n and in Chapter II. with ϵ^{ax} and $\sin ax$, and unless one is really intending to make a rather complete study of the Calculus, nothing further is needed. Our knowledge of those three functions is sufficient for nearly every practical engineering purpose. It will be found, indeed, that many of the examples given in this chapter might have been given in Chapters I. and II. For the differentiation and integration of functions in general, we should have preferred to ask students to read the regular treatises, skipping difficult parts in a first reading and afterwards returning to these parts when there is the knowledge which it is necessary to have before one can understand them. If a student has no tutor to mark these difficult parts for him, he will find them out for himself by trial.

By means of a few rules it is easy to become able to differentiate any algebraic function of x, and in spite of our wish that students should read the regular treatises we are weak enough to give these rules here. They are mainly used to enable schoolboys to prepare for examinations and attain facility in differentiation. These boys so seldom learn more of this wonderful subject, and so rapidly lose the facility in question, because they never have learnt really what $\frac{dy}{dx}$ means, that we are apt with beginners to discourage much practice in differentiation, and so err, possibly, as much as the older teachers, but in another way. If, however, a man sees clearly the object of his work, he ought to try to gain this facility in differentiation and to retain it. The knack is easily learnt, and in working the examples he will, at all

events, become more expert in manipulating algebraic and trigonometric expressions, and such expertness is all-important to the practical man.

In Chapters I. and II. we thought it very important that students should graph several illustrations of

$$y=ax^n,\ y=a\epsilon^{bx},\ y=a\sin(bx+c).$$

So also they ought to graph any new function which comes before them. But we would again warn them that it is better to have graphed a few very thoroughly, than to have a hazy belief that one has graphed a great number.

The engineer discovers himself and his own powers in the first problem of any kind that he is allowed to work out completely by himself. The nature of the problem does not matter; what does matter is the thoroughness with which he works it out.

Graph $y=\tan ax$. We assume that the student has already graphed $y=a\epsilon^{bx}\sin nx$.

194. If $y=f(x)$, so that when a particular value of x is chosen, y may be calculated; let a new value of x be taken, $x+\delta x$, this enables us to calculate the corresponding value of y,

or $$y+\delta y=f(x+\delta x).$$

Now subtract and divide by δx, and we find

$$\frac{\delta y}{\delta x}=\frac{f(x+\delta x)-f(x)}{\delta x}\ \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

We are here indicating, generally, what we must do with any function, and what we have already done with our famous three, and we see that **our definition of dy/dx** is, the limiting value reached by (1) as δx is made smaller and smaller without limit.

195. It is evident from this definition that the differential coefficient of $af(x)$, is a multiplied by the differential coefficient of $f(x)$, and it is easy to show that the differential coefficient of a **sum** of functions is equal to the sum of the differential coefficients of each. In some of the examples of Chapter I. we have assumed this without proof.

We may put the proof in this form:—

Let $y = u + v + w$, the sum of three given functions of x. Let x become $x + \delta x$,† let u become $u + \delta u$, v become $v + \delta v$, and w become $w + \delta w$. It results that if y becomes $y + \delta y$, then

$$\delta y = \delta u + \delta v + \delta w,$$

and
$$\frac{\delta y}{\delta x} = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta x} + \frac{\delta w}{\delta x},$$

and in the limit
$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx}.$$

196. *Differential Coefficient of a* **Product** *of two Functions.*

Let $y = uv$ where u and v are functions of x. When x becomes $x + \delta x$, let

$$y + \delta y = (u + \delta u)(v + \delta v) = uv + u \,.\, \delta v + v \,.\, \delta u + \delta u \,.\, \delta v.$$

Subtracting we find

$$\delta y = u \,.\, \delta v + v \,.\, \delta u + \delta u \,.\, \delta v,$$

and
$$\frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \frac{\delta u}{\delta x} \,.\, \delta v.$$

We now imagine δx, and in consequence (for this is always assumed in our work) δu, δv and δy to get smaller and smaller without limit. Consequently, whatever $\frac{du}{dx}$ may be, $\frac{du}{dx} \,.\, \delta v$ must in the limit become 0, and hence

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

The student must translate this for himself into ordinary language. It is in the same way easy to show, by writing uvw as $uv \times w$, that if $y = uvw$ then

$$\frac{dy}{dx} = uv\frac{dw}{dx} + vw\frac{du}{dx} + wu\frac{dv}{dx}.$$

Illustrations. If $y = 10x^7$ then, directly, $\frac{dy}{dx} = 70x^6$. But we may write it $y = 5x^3 \times 2x^4$.

Our new rule gives

$$\frac{dy}{dx} = 5x^3\,(8x^3) + 2x^4\,(15x^2) = 40x^6 + 30x^6 = 70x^6.$$

The student ought to manufacture other examples for himself.

197. *Differential Coefficient of a* **Quotient.**

Let $y = \frac{u}{v}$ when u and v are functions of x.

Then $$y + \delta y = \frac{u + \delta u}{v + \delta v}.$$

Subtract and we find

$$\delta y = \frac{u + \delta u}{v + \delta v} - \frac{u}{v} = \frac{v\,.\,\delta u - u\,.\,\delta v}{v^2 + v\,.\,\delta v},$$

$$\frac{\delta y}{\delta x} = \frac{v\,\frac{\delta u}{\delta x} - u\,\frac{\delta v}{\delta x}}{v^2 + v\,.\,\delta v}.$$

Letting δx get smaller and smaller without limit, $v\,.\,\delta v$ becomes 0, and we have

$$\frac{dy}{dx} = \frac{v\,\frac{du}{dx} - u\,\frac{dv}{dx}}{v^2}.$$

Here again the student must translate the rule into ordinary language, and he must get very well used indeed to the idea that it is $v\,\frac{du}{dx}$ which comes first:—

Denominator into differential coefficient of numerator, minus numerator into differential coefficient of denominator, divided by denominator squared.

A few illustrations ought to be manufactured. Thus $y = \frac{24x^7}{3x^2}$ is really $8x^5$, and $\frac{dy}{dx} = 40x^4$.

By our rule, $\frac{dy}{dx} = \frac{3x^2\,(168x^6) - 24x^7\,(6x)}{9x^4} = 40x^4.$

The student ought to work a few like $y=\frac{15x^2}{3x^4}=5x^{-2}$ or again $y=\frac{7x^{\frac{3}{2}}}{-2x^4}=-\frac{7}{2}x^{-\frac{5}{2}}$, and verify for himself.

198. If y is given as a function of z, and z is given as a function of x, then it is easy to express y as a function of x. Thus if $y=b\log(az^2+g)$ and $z=c+dx+\sin ex$, then

$$y=b\log\{a(c+dx+\sin ex)^2+g\}.$$

Now under such circumstances, that is, $y=f(z)$ and $z=F(x)$, if for x we take $x+\delta x$, and so calculate $z+\delta z$, and with this *same* $z+\delta z$ we calculate $y+\delta y$, then we can say that our δy is in consequence of our δx, and

$$\frac{\delta y}{\delta x}=\frac{\delta y}{\delta z}\times\frac{\delta z}{\delta x}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

This is evidently true because we have taken care that the two things written as δz shall be the same thing. On this supposition, that the two things written as δz remain the same however small they become, we see that the rule (1) is true even when δx is made smaller and smaller without limit, and as we suppose that δz also gets smaller and smaller without limit,

$$\mathbf{\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

This is such an enormously important proposition that a student ought not to rest satisfied until he sees very clearly that it is the case. For we must observe that the symbol dz cannot stand by itself; we know nothing of dz by itself; we only know of the complete symbols dy/dz or dz/dx.

We are very unwilling to plague a beginner, but it would be fatal to his progress to pass over this matter too easily. Therefore he ought to illustrate the law by a few examples. Thus let $y=az^3$ and $z=bx^2$. As $\frac{dy}{dz}=3az^2$, $\frac{dz}{dx}=2bx$, we have $\frac{dy}{dz}\cdot\frac{dz}{dx}=6abz^2x$ or $6ab^3x^5$. But by substitution, $y=ab^3x^6$, and if we differentiate directly we get the same answer. A student ought to manufacture many examples for himself.

An ingenious student might illustrate (2) by means of three curves, one connecting z and x, the other connecting z and y and a third produced by measurements from the other two, and by means of them show that for any value of x the slope of the y, x curve is equal to the product of the slopes of the other two. But in truth the method is too complex to be instructive. By an extension of our reasoning we see that

$$\frac{dy}{dx} = \frac{dy}{dw} \cdot \frac{dw}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} \quad \text{.................(3).}$$

199. It is a much easier matter to prove that

$$\mathbf{\frac{dy}{dx} \times \frac{dx}{dy} = 1} \quad \text{..... ....(4),}$$

by drawing a curve, because it is easy to see that $\frac{dx}{dy}$ is the cotangent of the angle of which $\frac{dy}{dx}$ is the tangent.

Otherwise:—if by increasing x by δx we obtain the increment δy of y, and if we take this same δy, so found, we ought to be able to find by calculation the very same δx with which we started. Hence

$$\frac{\delta y}{\delta x} \times \frac{\delta x}{\delta y} = 1 \quad \text{........................(5).}$$

On this proviso, however small δx may become, (5) is true and therefore (4) is true.

200. To illustrate (2). If a **gas engine indicator diagram** is taken, it is easy to find from it by applying Art. 57, a diagram for h, the rate at which the stuff shows that it is receiving heat in foot-pounds per unit change of volume, on the assumption that it is a perfect gas receiving heat from some furnace. (In truth it is its own furnace; the heat comes from its own chemical energy.) Just as pressure is $\frac{dW}{dv}$, the rate at which work is done **per unit change of volume**; so h is $\frac{dH}{dv}$. Observe that h is in the same units as p, and to draw the curve for h it is not necessary to pay any attention to the scales for either p or v. They

may be measured as inches on the diagram. We know of no better exercise to bring home to a student the meaning of a differential coefficient, than to take the indicator diagram, enlarge it greatly, make out a table of many values of p and v, and find approximately $\frac{dp}{dv}$ for each value of v. This is better than by drawing tangents to the curve. Using these values, and having found the values of h or $\frac{dH}{dv}$ at every place, suppose we want to find the rate **per second** at which the stuff is receiving heat. If t represents time, $\frac{dH}{dt}=\frac{dH}{dv}\cdot\frac{dv}{dt}$, and hence it is only necessary to multiply h by $\frac{dv}{dt}$.

As $\frac{dv}{dt}$ is represented by the velocity of the piston, and as the motion of the piston is, as a first approximation, simple harmonic, we describe a semicircle upon the distance on the diagram which represents the stroke, and the ordinates of the semicircle represent $\frac{dv}{dt}$. We have therefore to multiply every value of h by the corresponding ordinate of the semicircle, and we obtain, to a scale easily determined, the diagram which shows at every instant $\frac{dH}{dt}$.

Having seen that $\frac{dy}{dx}=\frac{dy}{dv}\cdot\frac{dv}{dx}$ and that $\frac{dy}{dx}=1\div\frac{dx}{dy}$, we shall often treat dx or dy as if it were a real algebraic quantity, recollecting however that although dy or dx may appear by itself in an expression, it is usually only for facility in writing that it so appears; thus the expression

$$M\,.\,dx+N\,.\,dy=0\dots\dots\dots\dots\dots\dots(1),$$

may appear, where M and N are functions of x and y; but this really stands for

$$M+N\frac{dy}{dx}=0\ \dots\dots\dots\dots\dots\dots\dots(2).$$

Again, if $y=ax^2$, we may write

$$dy=2ax\,.\,dx\dots\dots\dots\dots\dots\dots\dots\dots(3),$$

but this only stands for $\dfrac{dy}{dx} = 2ax$ ....(4).

Our main reason for doing it is this, that if we wish to integrate (3) we have only to write in the symbol $\int$, whereas, if we wish to integrate (4) we must describe the process in words, and yet the two processes are really the same. We have already used dx and dy in this way in Chap. I.

Mere mathematical illustrations of Art. 198 may be manufactured in plenty. But satisfying food for thought on the subject, is not so easy to find. The law is true; it is not difficult to prove it; but the student needs to make the law part of his mental machinery, and this needs more than academic 'proof.'

Let us now use these principles.

201. Let $y = \mathbf{log\, x}$; this statement is exactly the same as $x = \epsilon^y$. Hence $\dfrac{dx}{dy} = \epsilon^y = x$ and $\dfrac{dy}{dx} = \dfrac{1}{x}$. We used the idea that the integral of x^{-1} is $\log x$, in Chap. I., without proof. It is the exceptional case of the integration of x^n.

202. If the differential coefficient of $\sin x$ is known to be $\cos x$, find the differential coefficient of $\sin ax$.

$$y = \mathbf{sin\, ax} = \sin u \text{ if } u = ax,$$

$$\frac{dy}{du} = \cos u \text{ and } \frac{du}{dx} = a,$$

so that
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos u \times a = a \cos ax.$$

Find the differential coefficient of $y = \mathbf{cos\, ax}$, knowing that the differential coefficient of $\sin x$ is $\cos x$,

$$y = \cos ax = \sin\left(ax + \frac{\pi}{2}\right) = \sin u \text{ say, where } \frac{du}{dx} = a,$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos u \times a = a \cos\left(ax + \frac{\pi}{2}\right) = -a \sin ax.$$

203. Let $y = \log (x + a)$.

Assume $x + a = u$, or $y = \log u$, then $\dfrac{du}{dx} = 1$ and $\dfrac{dy}{du} = \dfrac{1}{u}$,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} = \frac{1}{x+a}.$$

204. $y = \tan x$. Treat this as a quotient, $y = \dfrac{\sin x}{\cos x}$,

$$\frac{dy}{dx} = \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x}.$$

The student ought to work this example in a direct manner also.

205. $y = \cot x$. We now have choice of many methods. Treat this as a quotient, $y = \dfrac{\cos x}{\sin x}$,

$$\frac{dy}{dx} = \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x} = -\frac{1}{\sin^2 x},$$

or we might have treated it in this way,

$$y = u^{-1} \text{ if } u = \tan x,$$

$$\frac{dy}{dx} = -u^{-2} \times \frac{du}{dx} = -u^{-2} \times \frac{1}{\cos^2 x}$$

$$= -\frac{1}{\tan^2 x} \cdot \frac{1}{\cos^2 x} = -\frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x.$$

206. Let $y = \sin ax^2$, say $y = \sin u$, and $u = ax^2$.

Then $$\frac{du}{dx} = 2ax,$$

and $$\frac{dy}{du} = \cos u,$$

so that $$\frac{dy}{dx} = \cos u \times 2ax = 2ax \cos ax^2.$$

Let $y = \epsilon^{a \sin x}$, say $y = \epsilon^u$, and $u = a \sin x$, so that

$$\frac{dy}{du} = \epsilon^u, \quad \frac{du}{dx} = a \cos x,$$

so that $$\frac{dy}{dx} = \epsilon^u a \cos x, \text{ or } a \cos x \cdot \epsilon^{a \sin x}$$

207. $y = \sec x$. We may either treat this as a quotient, or as follows; $y = (\cos x)^{-1} = u^{-1}$ if $u = \cos x$.

$$\frac{du}{dx} = -\sin x,\ \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -u^{-2}(-\sin x)$$

$$= \frac{\sin x}{\cos^2 x} = \sec x \cdot \tan x.$$

208. In Art. 11 the equation to the cycloid was given in terms of an auxiliary angle ϕ; $x = a\phi - a\sin\phi$, $y = a - a\cos\phi$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at any point.

Here $\frac{dy}{dx} = \frac{dy}{d\phi} \cdot \frac{d\phi}{dx} = \frac{\mathbf{dy}}{\mathbf{d\phi}} \div \frac{\mathbf{dx}}{\mathbf{d\phi}}$

$$= a\sin\phi/(a - a\cos\phi) = \frac{\sin\phi}{1-\cos\phi}.$$

Also $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{d\phi}\left(\frac{dy}{dx}\right) \times \frac{d\phi}{dx}$$

$$= \frac{(1-\cos\phi)\cos\phi - \sin\phi(\sin\phi)}{(1-\cos\phi)^2} \div (a - a\cos\phi)$$

$$= \frac{-1}{a(1-\cos\phi)^2} = -\frac{a}{y^2}.$$

209. If $$x^2 + y^2 = a^2 \quad \text{......................(1)},$$

$$2x + 2y\frac{dy}{dx} = 0 \text{ or } \frac{dy}{dx} = -\frac{x}{y} \quad \text{...........(2)}.$$

If we want $\frac{dy}{dx}$ in terms of x only we must find y from (1) and use it in (2). But for a great many purposes (2) is useful as it stands.

In the same way, if $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0 \text{ or } \frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y}.$$

Again, if $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,\ \frac{dy}{dx} = \frac{b^2}{a^2}\frac{x}{y}.$$

Also if $$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}},$$

$$\tfrac{2}{3}x^{-\frac{1}{3}} + \tfrac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0 \text{ or } \frac{dy}{dx} = -\sqrt[3]{\frac{y}{x}}.$$

If $$y = \tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x,\ \frac{dy}{dx} = \cos^4 x.$$

If $$y = \tfrac{1}{3}\tan^3 x + \tan x,\ \frac{dy}{dx} = \sec^4 x.$$

Let $y = \sqrt{x^2 + a^2} = u^{\frac{1}{2}}$ if $u = x^2 + a^2$, $\dfrac{du}{dx} = 2x$, so that

$$\frac{dy}{dx} = \tfrac{1}{2}u^{-\frac{1}{2}} \times 2x \text{ or } \frac{x}{\sqrt{x^2 + a^2}}.$$

210. Let $y = \sin^{-1} \mathbf{x}$. In words, y is the angle whose sine is x. Hence $x = \sin y$,

$$\frac{dx}{dy} = \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}.$$

Hence $$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}.$$

We have extracted a square root, and our answer may be + or −. We must give to $\dfrac{dy}{dx}$ the sign of $\cos y$.

211. Similarly if $y = \cos^{-1} \mathbf{x}$,

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}.$$

212. Let $y = \tan^{-1} \mathbf{x}$, so that $x = \tan y$,

$$\frac{dx}{dy} = \frac{1}{\cos^2 y} = 1 + \tan^2 y = 1 + x^2,$$

$$\frac{dy}{dx} = \frac{1}{1 + x^2}.$$

213. Similarly if $y = \cot^{-1} \mathbf{x}$, then $\dfrac{dy}{dx} = -\dfrac{1}{1 + x^2}$.

214. It will be seen that (2) and (4) of Arts. 198 and 199 give us power to differentiate any ordinary expression, and students ought to work many examples. They ought to verify the list of integrals given at the end of the book. A student ought to keep by him a very complete list of integrals. He cannot hope to remember them all. Sometimes it is advisable to take logarithms of both sides before differentiating, as in the following case:

$y = x^x$. Here $\log y = x \log x$,

$$\frac{1}{y} \cdot \frac{dy}{dx} = x \times \frac{1}{x} + \log x,$$

$$\frac{dy}{dx} = x^x (1 + \log x). \dagger$$

215. In the following examples, letters like x, y, z, v, w, θ, &c. are used for the variables; letters like a, b, c, m, n, &c. are supposed constant. A student gets too familiar with x and y. Let him occasionally change x into t or θ or v, and change y also, before beginning to differentiate. He ought to test the answer of every integral by differentiation.

List of Fundamental Cases.

$$\frac{d}{dx} x^n = nx^{n-1}, \qquad \int x^m . dx = \frac{1}{m+1} x^{m+1};$$

$$\frac{d}{dx} (\log x) = \frac{1}{x}, \qquad \int \frac{1}{x} . dx = \log x;$$

$$\frac{d}{dx}(\sin mx) = m \cos mx, \qquad \int \cos mx . dx = \frac{1}{m} \sin mx;$$

$$\frac{d}{dx}(\cos mx) = -m \sin mx, \qquad \int \sin mx . dx = -\frac{1}{m} \cos mx;$$

$$\frac{d}{dx}(\tan ax) = \frac{a}{\cos^2 ax}, \qquad \int \frac{dx}{\cos^2 ax} = \frac{1}{a} \tan ax;$$

$$\frac{d}{dx}(\cot ax) = -\frac{a}{\sin^2 ax}, \qquad \int \frac{dx}{\sin^2 ax} = -\frac{1}{a} \cot ax;$$

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}, \qquad \int\frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a};$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}, \qquad \int\frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a};$$

$$\frac{d}{dx}(a^x) = a^x \log a, \qquad \int a^x \, . \, dx = \frac{a^x}{\log a}.$$

Many integrals that at first sight look different are really those given above. Even the use of $\sqrt{\ }$ or $\sqrt[3]{\ }$ instead of the numerical symbol of power or root, disguises a function to a beginner. Thus

$$\frac{1}{a\sqrt[3]{x}} \text{ is } \frac{1}{a}x^{-\frac{1}{3}},$$

and its integral is

$$\frac{1}{a}\left(\frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1}\right) \text{ or } \frac{3}{2a}x^{\frac{2}{3}}.$$

216. In some of the following integrals certain **substitutions** are suggested. The student must not be discouraged if he cannot see why these are suggested; these suggestions are the outcome of, perhaps, weeks of mental effort by some dead and gone mathematician. Indeed, some of them are no better than this, that we are told the answer and are merely asked to test if it is right by differentiation.

Just here, in learning the knack of differentiation and integration, the student who has a tutor for a few lessons has a great advantage over a student who works by himself from a book. Nevertheless the hardworking student who has no tutorial help has some advantages; what he learns he learns well and does not forget. The man who walks through England has some advantages over the man who only takes railway journeys. In learning to bicycle, I think that on the whole, it is better to be held on for the first few days; learning the knack of differentiation and integration is not unlike learning to bicycle.

Exercises and Examples.

1. $y = x \log x$, $\dfrac{dy}{dx} = 1 + \log x$.

2. $y = a\sqrt{x},\ \dfrac{dy}{dx} = \dfrac{a}{2\sqrt{x}}.$

3. $y = \log(\tan x),\ \dfrac{dy}{dx} = \dfrac{2}{\sin 2x}.$

4. $y = \dfrac{1-\tan x}{\sec x},\ \dfrac{dy}{dx} = -(\sin x + \cos x).$

5. $y = \log(\log x),\ \dfrac{dy}{dx} = \dfrac{1}{x \log x}.$

6. $x = \epsilon^{at} \sin bt,\ \dfrac{dx}{dt} = \sqrt{a^2+b^2}\,.\,\epsilon^{at}\ \sin(bt+c),$

where $$\tan c = \frac{b}{a}.$$

We here use the simplification of Art. 116. The student will note that by page 235, θ (standing for d/dt), operating n times upon $\sin bt$, multiplies its amplitude by b^n and gives a lead of n right angles. He now sees that if θ operates n times upon $\epsilon^{at} \sin bt$, it multiplies by $(a^2+b^2)^{n/2}$ and produces a lead nc.

Thus $$\frac{d^2x}{dt^2} = (a^2+b^2)\,\epsilon^{at}\sin(bt+2c);$$

and $$\frac{d^3x}{dt^3} = (a^2+b^2)^{\frac{3}{2}}\epsilon^{at}\sin(bt+3c).$$

7. $p = 2\tan^{-1}\sqrt{\dfrac{1-\theta}{1+\theta}},\ \dfrac{dp}{d\theta} = -\dfrac{1}{\sqrt{1-\theta^2}}.$

8. $y = \log(\epsilon^z + \epsilon^{-z}),\ \dfrac{dy}{dz} = \dfrac{\epsilon^z - \epsilon^{-z}}{\epsilon^z + \epsilon^{-z}}.$

9. $y = \sqrt{x^3},\ \dfrac{dy}{dx} = \frac{3}{2}\sqrt{x}.$

10. $y = ax^2 + bx + c,\ \dfrac{dy}{dx} = 2ax + b.$

11. $v = 2t^3,\ \dfrac{dv}{dt} = 6t^2.$

12. $p = cv^{-1\cdot37}, \quad \frac{dp}{dv} = -1\cdot37 cv^{-2\cdot37}.$

13. $\int \epsilon^{av} \,.\, dv = \frac{1}{a}\epsilon^{av}.$

14. $\int av^{-1\cdot37}\, dv = -\frac{a}{\cdot37} v^{-0\cdot37}.$

15. $\int (at^2 + bt + c)\, dt = \frac{1}{3}at^3 + \frac{1}{2}bt^2 + ct + g.$

16. $\int \sqrt{x^3} \,.\, dx = \int x^{\frac{3}{2}} \,.\, dx = \frac{2}{5}x^{\frac{5}{2}}.$

17. $\int \frac{dt}{t^3}$ is $\int t^{-3} \,.\, dt = \frac{t^{-3+1}}{-3+1} = -\frac{1}{2}t^{-2}$ or $\frac{-1}{2t^2}$.

18. $\int \frac{dt}{\sqrt[3]{t}}$ is $\int t^{-\frac{1}{3}} \,.\, dt = \frac{t^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} = \frac{3}{2}t^{\frac{2}{3}}.$

19. $$\int \frac{dx}{m + nx^2} = \frac{1}{n}\int \frac{dx}{\frac{m}{n} + x^2} = \frac{1}{n}\,\frac{1}{\sqrt{\frac{m}{n}}}\tan^{-1}\frac{x}{\sqrt{\frac{m}{n}}}\;\dagger$$

or $$\frac{1}{\sqrt{mn}}\tan^{-1} x\sqrt{\frac{n}{m}}.$$

20. $\int \sqrt[3]{a+v} \,.\, dv.$ Here let $a + v = y$ so that $dv = dy$, and we have $\int y^{\frac{1}{3}} \,.\, dy = \frac{3}{4}y^{\frac{4}{3}} = \frac{3}{4}(a+v)^{\frac{4}{3}}.$

21. $\int \frac{t^3 \,.\, dt}{(t+a)^m}.$ Let $t + a = y,\ dt = dy,$

$$\int \frac{(y-a)^3}{y^m}\, dy = \int \frac{y^3 - 3ay^2 + 3a^2y - a^3}{y^m}\, dy$$

$$= \int (y^{3-m} - 3ay^{2-m} + 3a^2y^{1-m} - a^3y^{-m})\, dy$$

$$= \frac{y^{4-m}}{4-m} - 3a\frac{y^{3-m}}{3-m} + 3a^2\frac{y^{2-m}}{2-m} - a^3\frac{y^{1-m}}{1-m},$$

and in this it is easy to substitute $t + a$ for y.

22. $\int \frac{x \,.\, dx}{(a+bx)^{\frac{1}{3}}}$. Let $a+bx=y$ so that $b \,.\, dx = dy$,

$$\frac{1}{b^2}\int \frac{y-a}{y^{\frac{1}{3}}} \,.\, dy = \frac{1}{b^2}\left\{\int y^{\frac{2}{3}}dy - \int a y^{-\frac{1}{3}}dy\right\} = \frac{1}{b^2}\left(\tfrac{3}{5}y^{\frac{5}{3}} - \tfrac{3}{2}y^{\frac{2}{3}}\right)$$

$$= \frac{3}{b^2}\left\{\tfrac{1}{5}(a+bx)^{\frac{5}{3}} - \tfrac{1}{2}(a+bx)^{\frac{2}{3}}\right\}.$$

23. $\int \frac{t}{\sqrt{a^2-t^2}} \,.\, dt = -\sqrt{a^2-t^2}$, evidently.

24. $\int \frac{dx}{x-a}$. Let $x-a=y$, $dx=dy$

$$\int \frac{dy}{y} = \log y = \log (x-a).$$

25. Since $\frac{1}{x^2-a^2} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\{\log (x-a) - \log (x+a)\} = \frac{1}{2a}\log \frac{x-a}{x+a}.$$

Similarly $\int \frac{dx}{(x-a)(x-b)} = \frac{1}{a-b}\log \frac{x-a}{x-b}$.

26. If $x^2+2Ax+B$ has real factors, then $\int \frac{dx}{x^2+2Ax+B}$ is of the form just given.

But if there are no real factors, then the integral may be written $\int \frac{dx}{x^2+2Ax+A^2+B-A^2}$ and if $y=x+A$ and $a^2=B-A^2$ we have $\int \frac{dy}{y^2+a^2}$ which is $\frac{1}{a}\tan^{-1}\frac{y}{a}$.

27. $\int \tan x \,.\, dx = -\int \frac{-\sin x}{\cos x} dx$. This is our first example of a great class of integrals, where the numerator of a fraction is seen to be the differential coefficient of the denominator. Let $y=\cos x$, then $dy = -\sin x \,.\, dx$, so that the above integral is $-\int \frac{dy}{y}$, or $-\log y$, or $-\log (\cos x)$.

28. Let $f'(x)$ stand for the differential coefficient of $f(x)$, and we are asked to find $\int \frac{\mathbf{f'(x).dx}}{\mathbf{f(x)}}$. Let $f(x)=y$, then $f'(x).dx=dy$, so that the integral becomes

$$\int \frac{dy}{y} = \log y = \log f(x).$$

Hence, if the numerator of a fraction is seen to be the differential coefficient of the denominator, the answer is

$$\log(\text{denominator}).$$

29. $$\int \frac{2bx\,.\,dx}{a+bx^2} = \log(a+bx^2).$$

30. $$\int \frac{x\,.\,dx}{a+bx^2} = \frac{1}{2b}\int \frac{2bx\,.\,dx}{a+bx^2} = \frac{1}{2b}\log(a+bx^2).$$

31. Reduce $\int \frac{(m+nx)\,dx}{a+bx+cx^2}$ to a simpler form. If the numerator were $2cx+b$, the integral would come under our rule in Ex. 28. Now the numerator can be put in the shape

$$\frac{n}{2c}(2cx+b)+m-\frac{nb}{2c},$$

so we may write the integral as

$$n\int \frac{2cx+b}{a+bx+cx^2}\,dx + \left(m-\frac{nb}{2c}\right)\int \frac{dx}{a+bx+cx^2}$$
$$= \frac{n}{2c}\log(a+bx+cx^2) + \left(m-\frac{nb}{2c}\right)\int \frac{dx}{a+bx+cx^2}.$$

The latter integral is given in Example 26.

32. $$\int \frac{x+b}{a^2+x^2}\,.\,dx = \tfrac{1}{2}\int \frac{2x\,.\,dx}{a^2+x^2} + \int \frac{b\,.\,dx}{a^2+x^2}$$
$$= \tfrac{1}{2}\log(a^2+x^2) + \frac{b}{a}\tan^{-1}\frac{x}{a}.$$

33. $$\int \frac{\sin x\,.\,dx}{a+b\cos x} = -\frac{1}{b}\int \frac{-b\sin x\,.\,dx}{a+b\cos x} = -\frac{1}{b}\log(a+b\cos x).$$

34. $$\int \frac{dx}{x \,.\, \log x} = \int \frac{1 + \log x - \log x}{x \log x}\, dx$$
$$= \int \frac{(1 + \log x)\, dx}{x \log x} - \int \frac{dx}{x}$$
$$= \log (x \log x) - \log x$$
$$= \log x + \log (\log x) - \log x$$
$$= \log (\log x).$$

When expressions involve x^m and $(a + bx)^n$, try substituting $y = a + bx$ or $y = \frac{a}{x} + b$.

35. Thus $$\int \frac{dx}{(a + bx)^2} = -\frac{1}{b(a+bx)}.$$

36. $$\int \frac{x \,.\, dx}{(a + bx)^2} = \frac{1}{b^2}\left\{\log (a + bx) + \frac{a}{a + bx}\right\}.$$

37. $$\int \frac{dx}{x^2 (a + bx)} = -\frac{1}{ax} + \frac{b}{a^2} \log \frac{a + bx}{x}.$$

38. Again $$\int \frac{dx}{(a + bx^2)^{m+1}} = \frac{1}{2ma} \frac{x}{(a + bx^2)^m}$$
$$+ \frac{2m - 1}{2ma} \int \frac{dx}{(a + bx^2)^m}$$

and so we have **a formula of reduction.**

When expressions involve $\sqrt{a + bx}$ try $y^2 = a + bx$.

39. Thus $$\int \frac{x \,.\, dx}{\sqrt{a + bx}} = -\frac{2\,(2a - bx)}{3b^2} \sqrt{a + bx}.$$

40. $\int \frac{1}{x} \sqrt{1 + \log x} \,.\, dx.$ Try $y = 1 + \log x$.

Answer: $\frac{2}{3} (1 + \log x)^{\frac{3}{2}}$.

41. $\int \frac{dx}{\epsilon^x + \epsilon^{-x}}.$ Try $\epsilon^x = y$. Answer: $\tan^{-1} \epsilon^x$.

217. Integration by Parts. Since, if u and v are functions of x,

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx},$$
$$uv = \int u \,.\, dv + \int v \,.\, du,$$

or $$\int \mathbf{u}\,.\,\mathbf{dv} = \mathbf{uv} - \int \mathbf{v}\,.\,\mathbf{du} \quad \text{..............(1).}$$

We may write (1) as $\int u\,.\,\frac{dv}{dx}\,.\,dx = uv - \int v\,.\,\frac{du}{dx}\,.\,dx.$

By means of this formula, the integral $\int u\,.\,dv$ may be made to depend upon $\int v\,.\,du$.

42. Thus to find $\int x^n.\log x\,.dx$. Let $u = \log x$ and $\frac{dv}{dx} = x^n$, so that $v = \frac{x^{n+1}}{n+1}$. Formula (1) gives us $\frac{x^{n+1}}{n+1}\log x - \int \frac{x^n}{n+1}\,dx$, or $\frac{x^{n+1}}{n+1}\left(\log x - \frac{1}{n+1}\right)$.

43. $\int \mathbf{x}\,.\,\epsilon^{\mathbf{ax}}\,.\,dx.$

Let $u = x$; $\frac{dv}{dx} = \epsilon^{ax}$, so that $v = \frac{1}{a}\epsilon^{ax}$; then formula (1) gives us $\int x\,.\,\epsilon^{ax}\,.\,dx = \frac{1}{a}x\epsilon^{ax} - \frac{1}{a}\int \epsilon^{ax}\,.\,dx = \frac{1}{a}x\epsilon^{ax} - \frac{1}{a^2}\epsilon^{ax}$

$$= \frac{1}{a}\epsilon^{ax}\left(x - \frac{1}{a}\right).$$

44. $\int \epsilon^{\mathbf{ax}}\,.\,\mathbf{sin\,bx}\,.\,dx.$ Call the answer A.

Let $u = \sin bx$, $v = \frac{1}{a}\epsilon^{ax}$, then formula (1) gives us

$$A = \frac{1}{a}\epsilon^{ax}\sin bx - \frac{b}{a}\int \epsilon^{ax}\,.\cos bx\,.\,dx = \frac{1}{a}\epsilon^{ax}\sin bx - \frac{b}{a}B.$$

But similarly $\int \epsilon^{ax}\,.\cos bx\,.\,dx$, which we have called B, may be converted, if we take $u = \cos bx$ and $v = \frac{1}{a}\epsilon^{ax}$;

$$B = \frac{1}{a}\epsilon^{ax}\cos bx + \frac{b}{a}\int \epsilon^{ax}\sin bx\,.\,dx = \frac{1}{a}\epsilon^{ax}.\cos bx + \frac{b}{a}A.$$

Hence $A = \frac{1}{a} \epsilon^{ax} \sin bx - \frac{b}{a}\left(\frac{1}{a} \epsilon^{ax} \cos bx + \frac{b}{a} A\right)$, so that

$$A = \int \epsilon^{ax} \sin bx \,.\, dx = \frac{\epsilon^{ax} (a \sin bx - b \cos bx)}{a^2 + b^2}.$$

Similarly $B = \int \epsilon^{ax} \cos bx \,.\, dx = \frac{\epsilon^{ax} (a \cos bx + b \sin bx)}{a^2 + b^2}$.

218. By means of **Formulae of Reduction** we reduce integrals by successive steps to forms which are known to us. They are always deduced by the method of integration by parts. Thus

$$\int x^n \epsilon^{ax} \,.\, dx = \frac{1}{a} x^n \epsilon^{ax} - \frac{n}{a} \int x^{n-1} \,.\, \epsilon^{ax} \,.\, dx.$$

If then we have to integrate $x^4 \epsilon^{ax}$, we make it depend upon $x^3 \epsilon^{ax}$; again using this formula of reduction we make $x^3 \epsilon^{ax}$ depend upon $x^2 \epsilon^{ax}$, and so on, till we reduce to $x^0 \epsilon^{ax}$ or ϵ^{ax}, whose integral we know.

$$\text{Thus } \int x^3 \epsilon^x \,.\, dx = x^3 \epsilon^x - 3 \int x^2 \epsilon^x \, dx$$
$$= x^3 \epsilon^x - 3 \left\{ x^2 \epsilon^x - 2 \int x \epsilon^x \,.\, dx \right\}$$
$$= x^3 \epsilon^x - 3x^2 \epsilon^x + 6 \left(x \epsilon^x - \int \epsilon^x \,.\, dx \right)$$
$$= (x^3 - 3x^2 + 6x - 6)\, \epsilon^x.$$

Some General Exercises.

45. $y = a \sin^2 bx$, $\frac{dy}{dx} = ab \sin 2bx$.

46. $y = b \sin ax^n$, $\frac{dy}{dx} = bnax^{n-1} \cos ax^n$.

47. $y = (a + bx^n)^m$, $\frac{dy}{dx} = nbx^{n-1} m (a + bx^n)^{m-1}$.

48. $y = (a + bx)\, \epsilon^{cx}$, $\frac{dy}{dx} = \epsilon^{cx} (b + ac + bcx)$.

49. $y = a^x, \ \dfrac{dy}{dx} = a^x \,.\, \log a.$

50. $y = \log_a x, \ \dfrac{dy}{dx} = \dfrac{1}{x \log a}.$

51. $v = \dfrac{a - t}{t}, \ \dfrac{dv}{dt} = -\dfrac{a}{t^2}.$

52. $v = \sqrt{a^2 - t^2}, \ \dfrac{dv}{dt} = -\dfrac{t}{\sqrt{a^2 - t^2}}.$

53. $u = \dfrac{v^3}{(1 - v^2)^{\frac{3}{2}}}, \ \dfrac{du}{dv} = \dfrac{3v^2}{(1 - v^2)^{\frac{5}{2}}}.$

54. $v = \dfrac{\sqrt{a + t}}{\sqrt{a} + \sqrt{t}}, \ \dfrac{dv}{dt} = \dfrac{\sqrt{a}(\sqrt{t} - \sqrt{a})}{2\sqrt{t}\sqrt{(a + t)}(\sqrt{a} + \sqrt{t})^2}.$

55. $w = \sqrt{\dfrac{1 + y}{1 - y}}, \ \dfrac{dw}{dy} = 1 \div [\sqrt{1 - y^2} \,.\, (1 - y)].$

56. $y = \tan^{-1} \dfrac{2x}{1 - x^2}, \ \dfrac{dy}{dx} = \dfrac{2}{1 + x^2}.$

57. $y = \log(\sin x), \ \dfrac{dy}{dx} = \cot x.$

58. $u = \log \sqrt{a^2 - t^2}, \ \dfrac{du}{dt} = -\dfrac{t}{a^2 - t^2}.$

59. $y = \log \sqrt{\dfrac{1 - \cos t}{1 + \cos t}}, \ \dfrac{dy}{dt} = \dfrac{1}{\sin t}.$

60. $y = \sin^{-1} \dfrac{v}{\sqrt{1 + v^2}}, \ \dfrac{dy}{dv} = \dfrac{1}{1 + v^2}.$

61. $x = \tan^{-1} \dfrac{\sqrt{1 + t^2} + \sqrt{1 - t^2}}{\sqrt{1 + t^2} - \sqrt{1 - t^2}}, \ \dfrac{dx}{dt} = \dfrac{t}{\sqrt{1 - t^4}}.$

62. $x = \sec^{-1} t, \ \dfrac{dx}{dt} = \dfrac{1}{t\sqrt{t^2 - 1}}.$

63. $y = \sin(\log v), \ \dfrac{dy}{dv} = \dfrac{1}{v} \cos(\log v).$

64. $p=\sin^{-1}\dfrac{1-p^2}{1+p^2}$, $\dfrac{dv}{dp}=\dfrac{-2}{1+p^2}$.

65. $y=\dfrac{1+x}{1+x^2}$, $\dfrac{dy}{dx}=\dfrac{1-2x-x^2}{(1+x^2)^2}$.

66. $p=\log(\cot v)$, $\dfrac{dp}{dv}=-\dfrac{2}{\sin 2v}$.

67. $s=\epsilon^t(1-t^3)$, $\dfrac{ds}{dt}=\epsilon^t(1-3t^2-t^3)$.

68. $p=\dfrac{v^n}{(1+v)^n}$, $\dfrac{dp}{dv}=\dfrac{nv^{n-1}}{(1+v)^{n+1}}$.

69. $x=\dfrac{\epsilon^t-\epsilon^{-t}}{\epsilon^t+\epsilon^{-t}}$, $\dfrac{dx}{dt}=\dfrac{4}{(\epsilon^t+\epsilon^{-t})^2}$.

70. $p=\dfrac{\theta}{\epsilon^\theta-1}$, $\dfrac{dp}{d\theta}=\dfrac{\epsilon^\theta(1-\theta)-1}{(\epsilon^\theta-1)^2}$.

71. If $x=\tan\theta+\sec\theta$, prove that $\dfrac{d^2x}{d\theta^2}=\dfrac{\cos\theta}{(1-\sin\theta)^2}$.

72. If $x=\theta^2\log\theta$, prove that $\dfrac{d^3x}{d\theta^3}=\dfrac{2}{\theta}$.

73. If $y=\epsilon^{-x}\cos x$, prove that $\dfrac{d^4y}{dx^4}+4y=0$.

74. If $y=\dfrac{x^3}{1-x}$, prove that $\dfrac{d^4y}{dx^4}=\dfrac{24}{(1-x)^5}$.

75. $\int x^{m-1}(a+bx^n)^{p/q}\,dx$.

(1) If p/q be a positive integer, expand, multiply, and integrate each term.

(2) Assume $a+bx^n=y^q$; and if this fails,

(3) Assume $ax^{-n}+b=y^q$: this also may fail.

76. $\int x^2(a+x)^{\frac{1}{2}}\,.\,dx$. Let $a+x=y^2$, then $dx=2y\,.\,dy$, and $x=y^2-a$, so that we have $2\int(y^4-2ay^2+a^2)\,y^2\,.\,dy$, or $2\int(y^6-2ay^4+a^2y^2)\,dy$, which is easy.

77. $\int \frac{dx}{x^2(1+x^2)^{\frac{1}{2}}}$. Try $x^{-2}+1=y^2$, $\frac{1}{y^2-1}=x^2$, $-2x^{-3}.dx=2y.dy$ so that we have

$$=-\int dy=-y=-\sqrt{1+\frac{1}{x^2}}.$$

78. $\int \frac{dx}{(a^2+x^2)^{\frac{3}{2}}}$. Try $a^2x^{-2}+1=t^2$ and we find

$$-\frac{1}{a^2}\int\frac{dt}{t^2}=\frac{1}{a^2t}=\frac{x}{a^2\sqrt{a^2+x^2}}.$$

79. If $x=A\sin nt+B\cos nt$, prove that $\frac{d^2x}{dt^2}+n^2x=0$.

80. If $u=xy$, prove that $\frac{d^nu}{dx^n}=x\frac{d^ny}{dx^n}+n\frac{d^{n-1}y}{dx^{n-1}}$.

81. Illustrate the fact that $\frac{d^2u}{dy.dx}=\frac{d^2u}{dx.dy}$ (see Art. 83) in the following cases:

$$u=\tan^{-1}\frac{x}{y},\quad u=\sin(ax^n+by^n),$$

$$u=\sin(x^2y),\quad u=x\sin y+y\sin x,$$

$$u=bx^2\log ay,\quad u=\log\left(\tan\frac{y}{x}\right),$$

$$u=\frac{ay^2-bx}{by-ax^2},\quad u=xy\log(1+x^2y^2),$$

$$u=\frac{x^2y}{a^2-y^2}.$$

82. $y=\epsilon^{ax}\sin^m bx$, $\frac{dy}{dx}=\epsilon^{ax}\sin^{m-1}bx(a\sin bx+mb\cos bx)$.

83. $x=\epsilon^{-at}\cos bt$, $\frac{d^nx}{dt^n}=(a^2+b^2)^{\frac{n}{2}}\epsilon^{-at}\cos(bt-n\theta)$ where

$$\tan\theta=\frac{b}{a}.$$

84. $y=x^4\log x$, $\frac{d^6y}{dx^6}=-\frac{1.2.3.4}{x^2}$.

85. $y = \log(\sin x)$, $\dfrac{d^3y}{dx^3} = \dfrac{2\cos x}{\sin^3 x}$.

86. If $v = A_1 x^{a_1} y^{b_1} + A_2 x^{a_2} y^{b_2} + \&c.$, where

$$a_1 + b_1 = a_2 + b_2 = \&c. = n,$$

v is called a homogeneous function of x and y of n dimensions. Show that $x\left(\dfrac{dv}{dx}\right) + y\left(\dfrac{dv}{dy}\right) = nv$. Illustrate this when $v = \dfrac{xy}{x+y}$ and $v = \sqrt[3]{x^2 + y^2}$.

87. In general if $u = f(y + ax) + F(y - ax)$, where f and F are any functions whatsoever, prove that

$$\frac{d^2u}{dx^2} = a^2 \frac{d^2u}{dy^2},$$

the differentiation of course being partial.

88. If $u = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$, prove that $\dfrac{d^2u}{dx^2} + \dfrac{d^2u}{dy^2} + \dfrac{d^2u}{dz^2} = 0$.

89. If $s = a\epsilon^{-\alpha t}\sin\beta t$ satisfies $\dfrac{d^2s}{dt^2} + 2f\dfrac{ds}{dt} + n^2 s = 0$, find f and n^2 in terms of α and β, or find α and β in terms of f and n^2.

90. If $y = \epsilon^{\alpha x}$ is a solution of

$$\frac{d^4y}{dx^4} + A\frac{d^3y}{dx^3} + B\frac{d^2y}{dx^2} + C\frac{dy}{dx} + Dy = 0,$$

find α. As an example take

$$\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0,$$

and find its solution.

Answer: $y = a\epsilon^{x} + b\epsilon^{-x} + c\epsilon^{2x} + e$, where a, b, c, e are any constants whatsoever.

219. To integrate any fraction of the form

$$\frac{Ax^m + Bx^{m-1} + Cx^{m-2} + \&c.}{ax^n + bx^{n-1} + cx^{n-2} + \&c.} \quad \text{..............(1)},$$

where m and n are positive integers.

If m is greater than or equal to n, divide, and we have a quotient together with a remainder. The quotient is at once integrable and we have left a fraction of the form (1) in which m is less than n. Now the factors of the denominator can always be found and the fraction split up into partial fractions.

For every factor of the denominator of the shape $x-\alpha$ assume that we have a partial fraction $\frac{A}{x-\alpha}$; for every factor of the shape $x^2+ax+\beta$ assume that we have a partial of the shape $\frac{Ax+B}{x^2+\alpha x+\beta}$; if there are n equal factors each of them being $x-\alpha$ assume that we have the corresponding partial fractions

$$\frac{A_1}{(x-\alpha)^n}+\frac{A_2}{(x-\alpha)^{n-1}}+\&c.$$

Thus for example, suppose we have to deal with a fraction which we shall call $\frac{f(x)}{F(x)}$ and that $F(x)$ splits up into factors $x-\alpha$, $x-\beta$, x^2+ax+b, $(x-\gamma)^n$; we write

$$\frac{f(x)}{F(x)}=\frac{A}{x-\alpha}+\frac{B}{x-\beta}+\frac{Cx+D}{x^2+ax+b}+\frac{E}{(x-\gamma)^n}$$
$$+\frac{G}{(x-\gamma)^{n-1}}+\&c. \ldots\ldots(2).$$

Now multiply by $F(x)$ all across and we can either follow certain rules or we can exercise a certain amount of mother wit in finding A, B, C, D, E, F, G, &c.

Notice that as we have an *identity*, that is, an equation which is true for *any* value of x, it is true if we put $x=\alpha$ or $x=\beta$ or $x=\gamma$ or $x^2+ax+b=0$. Do all these things and we find that we have obtained A, B, E, C and D. To find G we may have first to differentiate our identity and then put $x=\gamma$ and so on. You will have found it more difficult to understand this description than to actually carry out the process.

Having split our given fraction into partials the integration is easy.

91. $$\frac{x^2}{(x-1)^2(x^2+1)} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1}.$$

Hence $x^2 = A(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x-1)^2$.

Let $x^2+1=0$, and we have with not much difficulty $C=-\frac{1}{2}$, $D=0$. Put $x=1$, and we have $A=\frac{1}{2}$. To find B, make $x=0$, and we find $B=\frac{1}{2}$. Hence we have to integrate

$$\frac{1}{2}\frac{1}{(x-1)^2} + \frac{1}{2}\frac{1}{x-1} - \frac{1}{2}\frac{x}{1+x^2},$$

and the answer is

$$-\frac{1}{2}\frac{1}{x-1} + \tfrac{1}{2}\log(x-1) - \tfrac{1}{4}\log(x^2+1).$$

When there are r equal quadratic factors, we assume the partials

$$\frac{C_1x+D_1}{(x^2+\alpha x+\beta)^r} + \frac{C_2x+D_2}{(x^2+\alpha x+\beta)^{r-1}} + \&c.$$

It is not difficult to see how all the constants are determined. We seldom, however, have complicated cases in our practical work.

92. Integrate $\dfrac{x^2+x-1}{x^3+x^2-6x}$ or $\dfrac{x^2+x-1}{x(x+3)(x-2)}$;

assume it to be equal to

$$\frac{M}{x} + \frac{N}{x+3} + \frac{P}{x-2};$$

so that $x^2+x-1 = M(x+3)(x-2) + Nx(x-2) + Px(x+3)$.

As this is true for all values of x, put $x=0$ and find M, put $x=-3$ and find N, put $x=2$ and find P. Thus we find that the given fraction splits up into

$$\frac{1}{6}\frac{1}{x} + \frac{1}{3}\frac{1}{x+3} + \frac{1}{2}\frac{1}{x-2};$$

so that the integral is

$$\tfrac{1}{6}\log x + \tfrac{1}{3}\log(x+3) + \tfrac{1}{2}\log(x-2).$$

93. $$\int \frac{5x^3+1}{x^2-3x+2}\,dx = \int\left(5x+15+\frac{35x-29}{x^2-3x+2}\right)dx$$
$$= \int\left(5x+15-\frac{6}{x-1}+\frac{41}{x-2}\right)dx$$
$$= \frac{5x^2}{2}+15x-6\log(x-1)+41\log(x-2).$$

94. $$\int \frac{x^5}{x^3-7x-6}\,dx$$
$$= \int\left(x^2+7+\frac{1}{4}\,\frac{1}{x+1}-\frac{32}{5}\,\frac{1}{x+2}+\frac{243}{20}\,\frac{1}{x-3}\right)dx.$$

95. $$\int \frac{x\,.\,dx}{x^3+x^2+x+1}$$
$$= \tfrac{1}{2}\tan^{-1}x+\tfrac{1}{4}\log(1+x^2)-\tfrac{1}{2}\log(1+x).$$

96. $$\frac{9x^2+9x-128}{x^3-5x^2+3x+9} = \frac{A}{x+1}+\frac{B_1}{(x-3)^2}+\frac{B_2}{x-3},$$

and we find $A=-8,\ B_1=-5,\ B_2=17$;

so that the integral is

$$-8\log(x+1)+\frac{5}{x-3}+17\log(x-3).$$

97. $$\int \frac{x\,.\,dx}{x^2+2x-3} = \tfrac{3}{4}\log(x+3)+\tfrac{1}{4}\log(x-1).$$

98. $$\int \frac{dx}{x^4+5x^2+4} = \tfrac{1}{3}\tan^{-1}x-\tfrac{1}{6}\tan^{-1}\frac{x}{2}.$$

99. $$\int \frac{(2x+3)\,dx}{x^3+x^2-2x} = -\tfrac{3}{2}\log x+\tfrac{5}{3}\log(x-1)-\tfrac{1}{6}\log(x+2).$$

100. $$\int \frac{x^2}{x^4-x^2-12}\,dx = \frac{\sqrt{3}}{7}\tan^{-1}\frac{x}{\sqrt{3}}+\tfrac{1}{7}\log\frac{x-2}{x+2}.$$

101. $$\int \frac{x\,.\,dx}{(1+x)(1+x^2)} = \tfrac{1}{2}\log\frac{1+x^2}{(1+x)^2}+\tfrac{1}{2}\tan^{-1}x.$$

102. $$\int \frac{x\,.\,dx}{x^2+2x-3} = \tfrac{3}{4}\log(x+3)+\tfrac{1}{4}\log(x-1).$$

103. $\displaystyle\int\frac{x^3 \,.\, dx}{x^2+6x+8}=\frac{x^2}{2}-6x+32\log(x+4)-4\log(x+2).$

104. $\displaystyle\int\frac{x \,.\, dx}{x^2-x-2}=\tfrac{2}{3}\log(x-2)+\tfrac{1}{3}\log(x+1).$

105. $\displaystyle\int\frac{(x-1)\,dx}{(x-3)(x+2)}=\tfrac{2}{5}\log(x-3)+\tfrac{3}{5}\log(x+2).$

106. $\displaystyle\int\frac{dx}{x^2+6x+8}=\tfrac{1}{2}\log\frac{x+2}{x+4}.$

107. $\displaystyle\int\frac{(2x-5)\,dx}{(x+3)(x+1)^2}=-\frac{7}{2(x+1)}+\tfrac{11}{4}\log\frac{x+1}{x+3}.$

108. $\displaystyle\int\frac{dx}{1+x+x^2}=\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}.$

109. $\displaystyle\int\frac{dx}{x^2+4x+3}=\tfrac{1}{2}\log\frac{x+1}{x+3}.$

110. $\displaystyle\int\frac{dx}{x^2+x-12}=\tfrac{1}{7}\log\frac{x-3}{x+4}.$

111. $\displaystyle\int\frac{dx}{x^2+4x+5}=\tan^{-1}(x+2).$

112. $\displaystyle\int\frac{dx}{1-2x+2x^2}=\tan^{-1}(2x-1).$

220. Maxima and Minima. If we draw any curve with maxima and minima points, and also draw the curve showing the value of $\frac{dy}{dx}$ in the first curve, we notice that;— where y is a maximum, $\frac{dy}{dx}=0$ and $\frac{d^2y}{dx^2}$ is *negative*; whereas, where y is a minimum, $\frac{dy}{dx}=0$ and $\frac{d^2y}{dx^2}$ is *positive.* If in any practical example we can find no easier way of discriminating, we use this way.

Notice, however, that what is here called a maximum value, means that y has gradually increased to that value and begins to diminish. y may have many maximum and

minimum values, the curve being wavy. Notice that $\frac{dy}{dx}$ may be 0 and $\frac{d^2y}{d^2x}=0$ so that there is neither a maximum nor a minimum value, y ceasing to increase and then beginning to increase again. See M, fig. 6.

1. Find the maximum and minimum values of $\frac{x}{x^2+1}$.

Answer: $\frac{1}{2}$ and $-\frac{1}{2}$.

2. Find the greatest value of $\frac{x}{(a^2+x)(b^2+x)}$.

Answer: $\frac{1}{(a+b)^2}$.

3. Prove that $a \sec\theta + b \operatorname{cosec}\theta$ is a minimum when

$$\tan\theta = \sqrt[3]{\frac{b}{a}}.$$

4. When is $\frac{1+3x}{\sqrt{4+5x^2}}$ a maximum? Answer: $x=\frac{12}{5}$.

5. When is $x^m(a-x)^n$ a maximum or minimum?

Answer: $x=\frac{ma}{m+n}$, a maximum.

6. Given the angle C of a triangle, prove that $\sin^2 A+\sin^2 B$ is a maximum and $\cos^2 A+\cos^2 B$ is a minimum when $A=B$.

7. $y=a\sin x+b\cos x$. What are the maximum and minimum values of y?

Answer: maximum is $y=\sqrt{a^2+b^2}$, minimum is $-\sqrt{a^2+b^2}$.

8. Find the least value of $a\tan\theta+b\cot\theta$.

Answer: $2\sqrt{ab}$.

9. Find the maximum and minimum values of

$$\frac{x^2+2x+11}{x^2+4x+10}.$$

Answer: 2 a maximum and $\frac{5}{6}$ a minimum.

Students ought to plot the function as a curve on squared paper.

10. Find the maximum and minimum values of

$$\frac{x^2 - x + 1}{x^2 + x - 1}.$$

Answer: maximum, -1.

11. Find the values of x which make $y = \dfrac{x^2 - 7x + 6}{x - 10}$ a maximum and a minimum.

Answer: $x = 4$ gives a maximum, $x = 16$ a minimum.

12. What value of c will make v a maximum if $v = \dfrac{1}{c} \log c$?

Answer: $c = \epsilon$.

13. If $p = \dfrac{(a+t)(b+t)}{t}$, $t = \sqrt{ab}$ gives a minimum value of p.

14. $x = \dfrac{\sin^3 \theta}{1 - \cos \theta}$, $\theta = \dfrac{\pi}{3}$ gives a maximum value to x.

15. What value of c will make v a minimum if

$$v = \frac{2}{1 + c - c^2}?$$

Answer: $c = \frac{1}{2}$.

16. When is $4x^3 - 15x^2 + 12x - 1$ a maximum or minimum?

Answer: $x = \frac{1}{2}$ a maximum; $x = 2$ a minimum.

17. $\tan^m x \tan^n (a - x)$ is a maximum when

$$\tan (a - 2x) = \frac{n - m}{n + m} \tan a.$$

18. $s = \dfrac{3t}{9 + t^2}$; $t = 3$ a maximum, $t = -3$ a minimum.

19. Given the vertical angle of a triangle and its area, find when its base is a minimum.

20. The characteristic of a series **Dynamo** is

$$E = \frac{aC}{1 + sC} \quad \text{.......................(1)},$$

where a is a number proportional to the angular velocity of the armature, and a and s depend upon the size of the iron, number of turns &c., E is the E.M.F. of the armature in volts and C the current in amperes. If r is the internal resistance of the machine in ohms and R is an outside resistance, the current

$$C = \frac{E}{r+R} \quad \text{.......................(2)},$$

and the power given out by the machine is

$$P = C^2 R \quad \text{.........................(3)}.$$

What value of R will make P a maximum?

Here (2) and (1) give $\dfrac{aC}{1+sC}\,\dfrac{1}{r+R} = C.$

So that $\quad 1 + sC = \dfrac{a}{r+R}, \quad C = \dfrac{1}{s}\left(\dfrac{a}{r+R} - 1\right),$

$$P = \frac{R}{s^2}\left(\frac{a}{r+R} - 1\right)^2, \text{ and if } \frac{dP}{dR} = 0,$$

we have $\left(\dfrac{a}{r+R} - 1\right)^2 + 2R\left(\dfrac{a}{r+R} - 1\right)\left(-\dfrac{a}{(r+R)^2}\right) = 0.$

Rejecting $\dfrac{a}{r+R} - 1 = 0$ because it gives $C = 0$, we have $\dfrac{a}{r+R} - 1 = \dfrac{2Ra}{(r+R)^2}$, and from this R may be found if r and a are given. Take $a = 1{\cdot}2$, $s = 0{\cdot}03$, $r = {\cdot}05$ and illustrate with curves.

21. A man is at sea 4 miles distant from the nearest point of a straight shore, and he wishes to get to a place 10 miles distant from this nearest point, the road lying along the shore. He can row and walk. Find at what point he ought to land, to get to this place in the minimum time, if he rows at 3 miles per hour and walks at 4 miles per hour. Assume that he can equally well leave his boat at one place as at another.

Fig. 100, $AC=4$, $CB=10$. Let him land at D where $CD=x$. Then $AD=\sqrt{16+x^2}$ and $DB=10-x$.

Hence the total time in hours $=\dfrac{\sqrt{16+x^2}}{3}+\dfrac{10-x}{4}$.

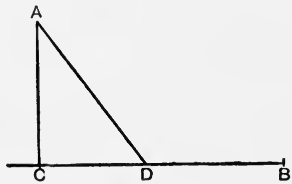

Fig. 100.

This is a minimum when $\frac{1}{3}x(16+x^2)^{-\frac{1}{2}}=\frac{1}{4}$, or $\frac{16}{9}x^2=16+x^2$, or $x=4{\cdot}535$ miles.

22. The candle power c of a certain kind of **incandescent lamp** $\times$ its probable life l in hours, was found experimentally to approximate on the average to

$$lc=10^{11{\cdot}697-0{\cdot}07545v},$$

where v is the potential difference in volts. The watts w expended per candle power were found to be

$$w=3{\cdot}7+10^{8{\cdot}007-{\cdot}07667v}.$$

The price of a lamp being 2*s.*, the lamps being lighted for 560 hours per year, and one electrical horse-power (or 746 watts) costing £2 for this year of 560 hours, find the most economical v for these lamps, so that the total cost in lamps and power may be a minimum.

$\dfrac{560}{l}$ lamps are needed per year, each costing £0·1. Cost per year is then $\dfrac{56}{l}$ in pounds, and this is for c candles, so that cost per year in pounds per candle is $\dfrac{56}{lc}$. Now £1 per year means $\dfrac{746}{2}$ watts, so that the cost per year per candle is

$$\frac{56}{lc}\times\frac{746}{2}\text{ watts.}$$

This added to w gives total cost in watts.

We have lc and w as functions of v. Hence

$$\frac{56 \times 746}{2} \cdot 10^{-11\cdot 697 + 0\cdot 07545 v} + 3\cdot 7 + 10^{8\cdot 007 - \cdot 07667 v}$$

is to be made a minimum.

Answer: $v = 101\cdot 15$ volts.

221. Sometimes when a particular value is given to x a function takes an **indeterminate form.** Thus for example in Art. 43, the area of the curve $y = mx^{-n}$ between the ordinates at $x = a$ and $x = b$ being $\int_a^b mx^{-n} \cdot dx$ was $\frac{m}{1-n}(b^{1-n} - a^{1-n})$.

Now when $n = 1$ the area becomes $\frac{m}{1-1}(1-1)$ or $\frac{0}{0}$, and this may obviously have any value whatsoever.

In any such case, say $\frac{f(x)}{F(x)}$, if $f(a) = 0$ and $F(a) = 0$, we proceed as follows. We take a value of x very near to a and find the limiting value of our expression as x is made nearer and nearer to a in value. Thus let $x = a + \delta x$.

Now as δx is made smaller and smaller it is evident that $f(x + \delta x)$ is more and more nearly $f(x) + \delta x \cdot \frac{df(x)}{dx}$†. If in this we put $x = a$, $f(x)$ or $f(a)$ disappears, and consequently our fraction $\frac{f(a+\delta x)}{F(a+\delta x)}$ becomes more and more nearly

$$\frac{\frac{d}{dx}f(x)}{\frac{d}{dx}F(x)}.$$

The rule then adopted is this:—Differentiate the numerator *only* and call it a new numerator; differentiate the denominator *only* and call it a new denominator; now insert the critical value of x, and we obtain the critical value of our fraction. The process may need repetition.

Example 1. Find the value of $\frac{\log x}{x-1}$ when $x = 1$.

First try, and we see that we have 0/0. Now follow the above rule, and we have $\frac{\frac{1}{x}}{1}$, and inserting in this $x=1$ we get 1 as our answer.

2. Find $\dfrac{ax^2-2acx+ac^2}{bx^2-2bcx+bc^2}$ when $x=c$.

First try $x=c$, and we get 0/0.

Now try $\dfrac{2ax-2ac}{2bx-2bc}$, and again we get 0/0.

Now repeating our process we get $\dfrac{a}{b}$.

3. Find $\dfrac{x-1}{x^n-1}$ when $x=1$. Answer: $\dfrac{1}{n}$.

4. Find $\dfrac{a^x-b^x}{x}$ when $x=0$. Answer: $\log\dfrac{a}{b}$.

5. Try the example referred to above. The area of a curve is $\dfrac{m}{1-n}(b^{1-n}-a^{1-n})=A$. If m, b, a are constants, what is the value of A when $n=1$? Writing it as

$$m\frac{b^{1-n}-a^{1-n}}{1-n},$$

differentiate both numerator and denominator *with regard to n*, and we have, since

$$\frac{d}{dn}(b^{1-n})=b^{1-n}\,.\log b\times(-1),$$

$$m\frac{b^{1-n}\log b-a^{1-n}\log a}{1},$$

and if we insert $n=1$ in this, we get

$$m(\log b-\log a)\ \text{ or }\ m\log\frac{b}{a},$$

which is indeed the answer we should have obtained if instead of taking our integral $\int x^{-1}\,.\,dx$ as following the rule

$$\int x^{-n}\,.\,dx=\frac{x^{-n+1}}{-n+1}+c,$$

we had remembered that in this special case

$$\int x^{-1} \,.\, dx = \log x.$$

222.† *Glossary and Exercises.*

Asymptote. A straight line which gets closer and closer to a curve as x or y gets greater and greater without limit. Thus $y = \frac{b}{a}\sqrt{x^2 - a^2}$ is a Hyperbola. Now as x gets greater and greater, so that $\frac{a}{x}$ is less and less important, the equation approaches more and more $y = \frac{b}{a}x$, which is the asymptote.

The test for an asymptote is that $\frac{dy}{dx}$ has a limiting value for points further and further from the origin, and the intercept of a tangent on the axis of x, $x - y\frac{dx}{dy}$, has a limiting value, or the intercept on the axis of y, $y - x\frac{dy}{dx}$, has a limiting value.

Point of Inflection. A point where $\frac{d^2y}{dx^2}$ changes sign.

Point of Osculation. A point where there are two or more equal values of $\frac{dy}{dx}$.

Cusp. Where two branches of a curve meet at a common tangent.

Conjugate Point. An isolated point, the coordinates of which satisfy the equation to the curve.

Point d'Arrêt. A point at which a single branch of a curve suddenly stops. *Example,* the origin in $y = x \log x$.

The **Companion to the Cycloid.** $x = a(1 - \cos\phi)$,
$y = a\phi$.

The **Epitrochoid.** $x = (a + b)\cos\phi - mb\cos\left(\frac{a}{b} + 1\right)\phi$,

$$y = (a + b)\sin\phi - mb\sin\left(\frac{a}{b} + 1\right)\phi,$$

where b is the radius of the rolling circle, a is the radius of the fixed circle, and mb = distance of tracing point along radius from centre of rolling circle. Make $m = 1$, and this is the Epicycloid.

The **Hypotrochoid.** $x = (a-b)\cos\phi + mb\cos\left(\frac{a}{b}-1\right)\phi,$

$$y = (a-b)\sin\phi - mb\sin\left(\frac{a}{b}-1\right)\phi.$$

Make $m = 1$, and we have the Hypocycloid.

Take $a = 4b$, and obtain a Hypocycloid in the form

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

Take $a = 2b$, and obtain the Hypocycloid which is a straight line.

In obtaining the Cycloid, Art. **11**, let the tracing point be anywhere on a radius of the rolling circle or the radius produced and obtain $x = a(1 - m\cos\phi)$, $y = a(\phi + m\sin\phi)$. If $m > 1$, or < 1, we have a prolate or a curtate Cycloid.

The **Lemniscata** $(x^2 + y^2)^2 = a^2(x^2 - y^2)$ becomes in polar coordinates $r^2 = a^2\cos 2\theta$,

and taking successively $\theta = 0$, $\theta = \cdot 1$, &c., we calculate r and graph the curve easily.

The Spiral of **Archimedes.** $r = a\theta$.

The Logarithmic or **Equiangular** Spiral. $r = a\epsilon^{b\theta}$.

The Logarithmic Curve. $y = a\log bx + c$.

The **Conchoid** $x^2y^2 = (a+x)^2(b^2 - x^2)$ becomes

$$r = a + b\sec\theta.$$

The **Cissoid** $y^2 = x^3/(2a - x)$ becomes $r = 2a\tan\theta \, . \sin\theta$.

The **Cardioide.** $r = a(1 - \cos\theta)$.

The **Hyperbolic** Spiral. $r\theta = a$.

The **Lituus** is $r^2\theta = a^2$.

The **Trisectrix.** $r = a(2\cos\theta \pm 1)$.

1. In the curve $y = \dfrac{a^2x}{a^2 + x^2}$, show that there are points of inflexion where x is 0 and $a\sqrt{3}$; the axis of x is an asymptote on both sides; there are points of maxima where $x = a$ and $-a$; the curve cuts the axis of x at 45°.

2. In $a^2y = 3bx^2 - x^3$ show that there is a point of inflexion where $x = b$, $y = \dfrac{2b^3}{a^2}$.

3. If $y^2x = 4a^2(2a - x)$, show that there are two points of inflexion when $x = \dfrac{3a}{2}$, $y = \pm \dfrac{2a}{\sqrt{3}}$.

4. If $y^2(x^2 - a^2) = x^4$, show that the equations to the asymptotes are $y = +x$ and $y = -x$.

5. The curve $x^3 - y^3 = a^3$ cuts the axis of x at right angles at $x = a$ where there is a point of inflexion.

6. Show that $y = a^2x/(ab + x^2)$ has three points of inflexion.

7. Prove again the statements of Exercise 2, Art. 99, and work the exercises there.

8. Find the subtangent and subnormal to the curve $y = \epsilon^{ax}$.

Answer: subtangent $\dfrac{1}{a}$, subnormal $a\epsilon^{2ax}$.

9. Find the subnormal and subtangent to the catenary.

$$y = \frac{c}{2}\left\{\epsilon^{\frac{x}{c}} + \epsilon^{-\frac{x}{c}}\right\}.$$

Answer: subnormal $= \dfrac{c}{4}\left\{\epsilon^{\frac{2x}{c}} - \epsilon^{-\frac{2x}{c}}\right\}$,

$$\text{subtangent} = c\,\frac{\epsilon^{\frac{x}{c}} + \epsilon^{-\frac{x}{c}}}{\epsilon^{\frac{x}{c}} - \epsilon^{-\frac{x}{c}}}.$$

10. Find the subtangent of the curve

$$x^3 - 3ayx + y^3 = 0,$$

$$3x^2 - 3ay - 3ax\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0.$$

Hence $$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}.$$

Subtangent at point x, y is $y\dfrac{dx}{dy} = y\dfrac{y^2 - ax}{ay - x^2}$.

11. In the curve $y^2 = \frac{x^3 + ax^2}{x - a}$ find the equation to the asymptote. Here $y^2 = x^2 \left(\frac{1 + \frac{a}{x}}{1 - \frac{a}{x}} \right) = x^2 \left(1 + \frac{2a}{x} + \frac{2a^2}{x^2} + \&c.\right)$ by division. As x is greater and greater, $\frac{2a}{x}$ &c. get smaller and smaller and in the limit (see Art. 3)

$$y = \pm x \left(1 + \frac{a}{x}\right),$$

$$y = \pm (x + a).$$

So we have a pair of asymptotes $y = x + a$, $y = -x - a$.

Again, the straight line $x = a$, a line parallel to the axis of y, is also an asymptote, y becoming greater and greater as x gets nearer and nearer to a in value.

12. Find the tangent to $y^{\frac{2}{3}} + x^{\frac{2}{3}} = a^{\frac{2}{3}}$.

$$\tfrac{2}{3} y^{-\frac{1}{3}} \frac{dy}{dx} + \tfrac{2}{3} x^{-\frac{1}{3}} = 0, \quad \frac{dy}{dx} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}.$$

Hence at the point x_1, y_1 the tangent is $\frac{y - y_1}{x - x_1} = -\sqrt[3]{\frac{y_1}{x_1}}$.

13. In the curve $y - 2 = (x - 1)\sqrt{x - 2}$, where is $\frac{dy}{dx} = \infty$? At what angle does the curve cut the axis?

$$\frac{dy}{dx} = \sqrt{x - 2} + (x - 1)\tfrac{1}{2}(x - 2)^{-\frac{1}{2}} = \frac{3x - 5}{2\sqrt{x - 2}}.$$

This is infinity where $x = 2$ and then $y = 2$; that is, the tangent at (2, 2) is at right angles to the axis of x.

Where $y = 0$, it will be found that $x = 3$ and $\frac{dy}{dx} = 2$.

14. In the curve $y^3 = ax^2 + x^3$, find the intercept by the tangent on the axis of y, that is, find $y - x\frac{dy}{dx}$.

$$3y^2 \frac{dy}{dx} = 2ax + 3x^2.$$

So that we want $y - x\dfrac{2ax+3x^2}{3y^2}$ or $\dfrac{3y^3-2ax^2-3x^3}{3y^2}$ and this will be found to be $\dfrac{a}{3}\left(\dfrac{x}{a+x}\right)^{\frac{2}{3}}$.

15. The length of the subnormal at x, y, is $2a^2x^3$, what is the curve ?

Here $y\dfrac{dy}{dx} = 2a^2x^3$. Hence $\frac{1}{2}y^2 = \frac{1}{2}a^2x^4$ or $y = ax^2$ is the equation to the curve, a parabola. The subtangent is $y\dfrac{dx}{dy}$, or $y \cdot \dfrac{y}{2a^2x^3}$, or $\dfrac{a^2x^4}{2a^2x^3}$, or $\frac{1}{2}x$.

16. Show that the length of the normal to the catenary† is $\dfrac{1}{c}y^2$.

17. Show that $y^4 - x^4 + 2bx^2y = 0$ has the two asymptotes $y = x - \dfrac{b}{2}$ and $y = -x - \dfrac{b}{2}$.

18. Show that the subtangent and subnormal to the circle $y^2 = 2ax - x^2$, are $\dfrac{2ax-x^2}{a-x}$ and $a - x$ respectively, and to the ellipse $y^2 = \dfrac{b^2}{a^2}(2ax - x^2)$ they are $\dfrac{2ax-x^2}{a-x}$ and $\dfrac{b^2}{a^2}(a-x)$.

19. Find the tangent to the cissoid $y^2 = \dfrac{x^3}{2a-x}$.

$$\text{Answer: } y = \left\{\frac{x}{(2a-x)^3}\right\}^{\frac{1}{2}} \{(3a-x)\,x_1 - ax\}.$$

20. What curve has a constant subtangent ?

$$y\frac{dx}{dy} = a \text{ or } dx = a\frac{dy}{y}, \text{ or } x = a\log y + c \text{ or } y = C\epsilon^{\frac{x}{a}},$$

the logarithmic curve.

21. Show that $x^3 - y^3 + ax^2 = 0$ has the asymptote

$$y = x + \frac{a}{3}.$$

22. Show that a curve is **convex or concave** to the axis of x as y and $\frac{d^2y}{dx^2}$ have the same or opposite signs. See Art. 60.

223. The circle which passes through a point in a curve, which has the same *slope* there as the curve, and which has also the same rate of change of slope, is said to be the circle of **curvature** there. If the centre of a circle has a and b for its co-ordinates, and if the radius is r, it is easy to see that its equation is

$$(x-a)^2+(y-b)^2=r^2 \qquad \text{.................(1).}$$

Differentiating (1) (and dividing by 2) and again differentiating we have

$$x-a+(y-b)\frac{dy}{dx}=0 \qquad \text{.................(2),}$$

and

$$1+(y-b)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0 \qquad \text{...........(3),}$$

writing p for $\frac{dy}{dx}$ and q for $\frac{d^2y}{dx^2}$ we have from (3)

$$y-b=-\frac{1+p^2}{q} \qquad \text{.................(4);}$$

using this in (2) we have

$$x-a=\frac{1+p^2}{q}p \qquad \text{....................(5).}$$

Now p and q and x and y at any point of the curve being known, we know that these are the same for the circle of curvature there, and so a and b can be found and also r. If the subject of evolutes were of any interest to engineers, this would be the place to speak of finding an equation connecting a and b, for this would be the equation of the evolute of the curve. The curve itself would then be called the involute to the evolute. Any practical man can work out this matter for himself. It is of more interest to find r the radius of curvature. Inserting (4) and (5) in (1) we find the curvature

$$\frac{1}{r}=\frac{q}{(1+p^2)^{\frac{3}{2}}} \qquad \text{......................(6).}$$

A better way of putting the matter is this:—A curve turns through the angle $\delta\theta$ in the length δs, and **curvature** is defined as the limiting value of

$$\frac{\delta\theta}{\delta s}, \text{ or } \frac{1}{r} = \frac{d\theta}{ds} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

Now $\tan\theta = \frac{dy}{dx} = p$, say, so that $\theta = \tan^{-1} p$. Hence

$$\frac{d\theta}{ds} = \frac{1}{1+p^2} \cdot \frac{dp}{ds}.$$

Now $\quad \frac{ds}{dx} = \sqrt{1+p^2} = \frac{ds}{dp} \cdot \frac{dp}{dx} = \frac{ds}{dp} \cdot \frac{d^2y}{dx^2}.$

Hence $\quad \frac{1}{r} = \frac{d\theta}{ds} = \frac{d^2y}{dx^2} \Big/ \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} \ldots\ldots\ldots\ldots\ldots(8).$

Exercises. 1. The equation to a curve is

$$x^3 - 1500x^2 + 30000x - 3000000y = 0.$$

Show that the denominator of $\frac{1}{r}$ in (8) is practically 1 from $x=0$ to $x=100$. Find the curvature where $x=0$.

2. In the curve $y = x^4 - 4x^3 - 18x^2$ find the curvature at the origin. Answer: 36.

3. Show that the radius of curvature at $x=a$, $y=0$ of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is $\frac{b^2}{a}$.

4. Find the radius of curvature where $x=0$, of the parabola, $y^2 = 4ax$. Answer: $r = 2a$.

5. Find the radius of curvature of $y = b\epsilon^{ax}$.

Answer: $q = a^2b\epsilon^{ax}$, $p = ab\epsilon^{ax}$, $r = \frac{(1+a^2b^2\epsilon^{2ax})^{\frac{3}{2}}}{a^2b\epsilon^{ax}}$, so that where $x=0$, $r = \frac{(1+a^2b^2)^{\frac{3}{2}}}{a^2b}$.

6. Find the radius of curvature of $y = a\sin bx$.

Answer: $r = \frac{(1+a^2b^2\cos^2 bx)^{\frac{3}{2}}}{-ab^2\sin bx}$. Where $x=0$, $r=\infty$, or curvature is 0; where $bx = \frac{\pi}{2}$, $r = \frac{1}{-ab^2}$.

7. Find the radius of curvature of the catenary

$$y = \frac{c}{2}\left(\epsilon^{\frac{x}{c}} + \epsilon^{-\frac{x}{c}}\right).$$

Answer: $r = \frac{y^2}{c}$. At the vertex where $x = 0$, $y = c$, $r = c$.

8. Show that the radius of curvature of

$$y^2(x - 4m) = mx(x - 3m),$$

at one of the points where $y = 0$, is $\frac{3m}{8}$, and at the other, $\frac{3m}{2}$.

9. Find the equation of the circle of curvature of the curve $y^4 = 4m^2x^2 - x^4$, where $x = 0$, $y = 0$.

10. The radius of curvature of $3a^2y = x^3$, is $r = \frac{(a^4 + x^4)^{\frac{3}{2}}}{2a^4x}$.

11. In the ellipse show that the radius of curvature is $(a^2 - e^2x^2)^{\frac{3}{2}} \div ab$, where $e^2 = 1 - \frac{b^2}{a^2}$, e being the eccentricity.

12. Find the radius of curvature of $xy = a$.

Answer: $(x^2 + y^2)^{\frac{3}{2}} \div 2a$.

13. Find the radius of curvature of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Answer: $(e^2x^2 - a^2)^{\frac{3}{2}} \div ab$, where $e^2 = 1 + \frac{b^2}{a^2}$.

14. In the catenary the radius of curvature is equal and opposite to the length of the normal.

15. Find the radius of curvature of the tractrix, the equation to which is $y\frac{dx}{dy} = \sqrt{a^2 - y^2}$.

224. Let $$f(x, y, a) = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$
be the equation to a family of curves, a being a constant for each curve, but called a **variable parameter** for the family,

as it is by taking different values for a that one obtains different members of the family. Thus

$$f(x, y, a + \delta a) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

is the next member of the family as δa is made smaller and smaller. Now (2) may be written (see (1) Art. 21)

$$f(x, y, a) + \delta a \,.\, \frac{d}{da} f(x, y, a) = 0 \ldots\ldots\ldots\ldots(3),$$

and the point of intersection of (2)† and (1) is obtainable by solving them as simultaneous equations in x and y; or again, if we eliminate a from (1) and

$$\frac{d}{da} f(x, y, a) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

we obtain a relation which must hold for the values of x and y, of the points of ultimate intersection of the curves formed by varying a continuously; this is said to be the equation of the **envelope** of the family of curves (1) and it can be proved that it is *touched* by every curve of the family.

Example. If by taking various values of a in

$$y = \frac{m}{a} + ax$$

we have a family of straight lines, find the envelope. Here $f(x, y, a) = 0$ is represented by

$$y - \frac{m}{a} - ax = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)^*,$$

and differentiating with regard to a we have

$$+\frac{m}{a^2} - x = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)^*,$$

or $$\frac{1}{x} = \frac{a^2}{m}, \text{ or } a^2 = \frac{m}{x}.$$

Using this in(1)* we have

$$y - \sqrt{mx} - x\sqrt{\frac{m}{x}} = 0,$$

or $$y - 2\sqrt{mx} = 0, \text{ or } y^2 = 4mx,$$

a parabola.

Example. In Ex. 3, Art. 24, if projectiles are all sent out with the same velocity V, at different angular elevations α, their paths form the family of curves,

$$y = \frac{V \sin \alpha}{V \cos \alpha} x - \tfrac{1}{2} g \frac{x^2}{V^2 \cos^2 \alpha},$$

or $$y - xa + mx^2 (a^2 + 1) = 0,$$

where a stands for $\tan \alpha$ and is a variable parameter, and

$$m = \frac{1}{2} \frac{g}{V^2}.$$

Differentiating with regard to a,

$$-x + 2mx^2 a = 0 \quad \text{or} \quad a = +\frac{1}{2mx};$$

$$\therefore \; y - \frac{1}{2m} + mx^2 \left(\frac{1}{4m^2 x^2} + 1\right) = 0$$

is the equation to the envelope, or

$$y = -mx^2 + \frac{1}{4m}.$$

This is the equation to a parabola whose vertex is $\frac{1}{4m}$ or $\frac{V^2}{2g}$ above the point of projection.

225.† **Polar Co-ordinates.** If instead of giving the position of a point P in x and y co-ordinates, we give it in terms of the distance OP called r, the *radius vector*, and the angle QOP (fig. 101) called θ, so that what we used to call x is $r \cos \theta$ and what we used to call y is $r \sin \theta$, the equations of some curves, such as spirals, become simpler. If the co-ordinates of P' are $r + \delta r$ and $\theta + \delta\theta$, then in the limit PSP' may be looked upon as a little right-angled triangle in which $PS = r \, . \, \delta\theta$, SP' is δr, PP' or $\delta s = \sqrt{r^2 (\delta\theta)^2 + (\delta r)^2}$ so that

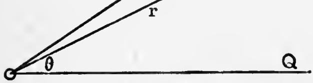

Fig. 101.

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}.$$

Also the elementary area POP' is in the limit $\frac{1}{2}r^2 . \delta\theta$, and the area enclosed between a radius vector at θ_1 and another at θ_2 is $\frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 . d\theta$, so that if r can be stated in terms of θ it is easy to find the area of the sector. Also the angle ϕ between the tangent at P and r is evidently such that $\tan\phi = PS/P'S$ or, $r\dfrac{d\theta}{dr} = \tan\phi$. This method of dealing with curves is interesting to students who are studying astronomy.

If $\mathbf{r = a^{b\theta}}$ (the equiangular spiral)

$$\frac{dr}{d\theta} = ba^{b\theta}\log a, \text{ and so, } r\frac{d\theta}{dr}, \text{ or } r \div \frac{dr}{d\theta} = 1/b\log a,$$

so that $\tan\phi$ is a constant; that is the curve everywhere makes the same angle with the radius vector.

Let $x = r\cos\theta$ so that x is always the projection of the radius vector on a line, $x = a^{b\theta}\cos\theta$. Now imagine the radius vector to rotate with uniform angular velocity of $-q$ radians per second starting with $\theta = 0$ when $t = 0$, so that $\theta = -qt$, then $x = a^{-bqt}\cos qt$.

Thus we see that if simple harmonic motion is the projection of uniform angular motion in a circle; **damped simple harmonic motion** is the projection of uniform angular motion in an equiangular spiral. See Note, Art. 112.

Ex. 1. Find the **area** of the curve $r = a(1 + \cos\theta)$. Draw the curve and note that the whole area is $\int_0^\pi r^2 . d\theta$, or $\frac{3}{2}\pi a^2$.

Ex. 2. Find the area of $r = a(\cos 2\theta + \sin 2\theta)$. Answer: πa^2.

Ex. 3. Find the area between the conchoid and two radii vectores. Answer:

$b^2(\tan\theta_2 - \tan\theta_1) + 2ab\log\{\tan(\pi/4 - \frac{1}{2}\theta_2) \div \tan(\pi/4 - \frac{1}{2}\theta_1)\}$.

226. *Exercises.* 1. Find the area of the surface generated by the revolution of the catenary (Art. 38) round the axis of y.

2. Prove that the equation to the cycloid, the vertex being the origin, is

$$x = a(\theta + \sin\theta) \qquad y = a(1 - \cos\theta),$$

if (fig. 102) $PB = x$, $PA = y$, $OCQ = \theta$.

Show that when the cycloid revolves about OY it generates a volume $\pi a^3\left(\frac{3\pi^2}{2}-\frac{8}{3}\right)$, and when it revolves about OX it

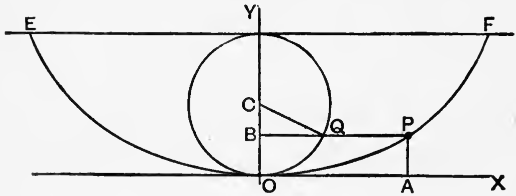

Fig. 102.

generates the volume $\pi^2 a^3$. If it revolves about EF it generates the volume $5\pi^2 a^3$.

3. Find the **length of the curve** $9ay^2 = 4x^3$.

Answer, $s = \int_0^x \sqrt{1+\frac{x}{a}} \,.\, dx = \frac{2}{3}a\left\{\left(1+\frac{x}{a}\right)^{\frac{3}{2}}-1\right\}$.

4. Find the length of the curve $y^2 = 2ax - x^2$.

Answer : $s = a \operatorname{vers}^{-1} x/a$.

5. Find the length of the cycloid. See Art. 47.

Answer : $s = 8a\,(1-\cos\frac{1}{2}\phi) = 8a - 4\sqrt{4a^2-2ay}$.

6. Find the length of the parabola $y=\sqrt{4ax}$, from the vertex.

Answer : $s = \sqrt{ax+x^2} + a\log\frac{\sqrt{x}+\sqrt{a}+x}{\sqrt{a}}$.

7. Show that the whole area of the companion to the cycloid is twice that of the generating circle.

8. Find the area of $r = b\epsilon^{\theta/c}$ between the radii r_1 and r_2, using $A = \int_{\theta_1}^{\theta_2}\frac{1}{2}r^2 \,.\, d\theta$.

Answer: $\frac{c}{4}(r_2{}^2 - r_1{}^2)$.

9. Show that in the logarithmic curve $x = a\epsilon^{\frac{y}{c}}$,

$$s = c \log \frac{x}{c + \sqrt{c^2 + x^2}} + \sqrt{c^2 + x^2} + C.$$

10. Show that in the curve $r = a(1 + \cos\theta)$, using

$$s = \int d\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2},$$

$$s = 4a \sin\frac{\theta}{2}.$$

11. Show that in the curve $r = b\epsilon^{\theta/c}$,

$$s = r\sqrt{1 + c^2} + C.$$

12. Show that in the cycloid,

$$\frac{dy}{ds} = \sqrt{1 - \frac{y}{2a}};$$

and consequently $\qquad s = 4\sqrt{a^2 - \tfrac{1}{2}ay} - 2a.$

13. Show that in the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$,

$$s = \tfrac{3}{2}a^{\frac{1}{3}}x^{\frac{2}{3}}.$$

14. The ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ revolves about the axis of x. Prove that the area of the surface generated is

$$2\pi ab\left\{\sqrt{1 - e^2} + \frac{\sin^{-1} e}{e}\right\},$$

where $\qquad e^2 = 1 - \frac{b^2}{a^2}.$

15. Show that the whole area of the curve, $\frac{x^{\frac{2}{3}}}{a^{\frac{2}{3}}} + \frac{y^{\frac{2}{3}}}{b^{\frac{2}{3}}} = 1$ is $\tfrac{3}{8}\pi ab$.

16. Find the area of the loop of the curve, $y = x\sqrt{\frac{a + x}{a - x}}$.

Answer: $2a^2\left(1 - \frac{\pi}{4}\right)$.

17. Find the whole area of $y = x + \sqrt{a^2 - x^2}$.

Answer: πa^2.

18. Find the area of a loop of the curve $r^2 = a^2 \cos 2\theta$.

Answer: $\frac{1}{2}a^2$.

19. Find the **area** of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; that is, find four times the value of the integral

$$\int_0^a \frac{b}{a} \sqrt{a^2 - x^2} \, . \, dx.$$

20. Find the **area** of the cycloid in terms of the angle ϕ (Art. 11).

$$\int y \, . \, dx = \int y \, . \, \frac{dx}{d\phi} \, . \, d\phi.$$

Answer: $a^2 (\frac{3}{2}\phi - 2 \sin \phi + \frac{1}{4} \sin 2\phi)$; and if the limits are $\phi = 0$ and $\phi = 2\pi$ we have the whole area equal to 3 times that of the rolling circle.

227. A **body** of weight W acted upon by gravity, **moves in a medium** in which the resistance $= av^n$, where v is the velocity and a and n are constants.

Then $\frac{W}{g} \frac{dv}{dt} = W - av^n$.

What is the velocity when acceleration ceases? Let v_1 be this **terminal velocity.** $av_1^n = W$, or our $a = Wv_1^{-n}$.

$$g \frac{dt}{dv} = \frac{1}{1 - \frac{av^n}{W}} = \frac{1}{1 - \left(\frac{v}{v_1}\right)^n};$$

so that
$$t = \frac{1}{g} \int \frac{dv}{1 - \left(\frac{v}{v_1}\right)^n}.$$

Thus let $n = 2$, $\quad t = \frac{v_1}{2g} \log \frac{v_1 + v}{v_1 - v}$,

or
$$v = v_1 \tanh \frac{gt}{v_1} = \frac{dx}{dt}.$$

If x is the depth fallen through,

$$x = \frac{v_1^2}{g} \log . \cosh \frac{gt}{v_1}.$$

228. Our old Example of Art. 24.

A point moves so that it has no acceleration horizontally and its acceleration downward is g a constant. Let y be measured *upwards* and x horizontally, then

$$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = -g,$$

$$\frac{dx}{dt} = c,$$

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = c \frac{dy}{dx},$$

$$\frac{d^2y}{dt^2} = c \cdot \frac{d^2y}{dx^2} \cdot \frac{dx}{dt} = c^2 \frac{d^2y}{dx^2}.$$

Hence $$\frac{d^2y}{dx^2} = \frac{-g}{c^2},$$

$$\frac{dy}{dx} = \frac{-g}{c^2} x + a,$$

$$y = -\frac{1}{2}\frac{g}{c^2} x^2 + ax + b \quad \text{.................(1)},$$

which is a parabola. Compare Art. 24.

If we take $y=0$ when $x=0$, $b=0$. Also we see that a is the tangent of the angle which the path makes with the horizontal at $x=0$ and c is the constant horizontal velocity. If a projectile has the initial velocity V with the upward inclination α, then $c = V \cos \alpha$, and $\tan \alpha = a$, so that (1) becomes

$$y = -\frac{1}{2} \frac{gx^2}{V^2 \cos^2 \alpha} + x \tan \alpha.$$

229. Exercises on Fourier.

1. A periodic function of x has the value $f(x) = mx$, from $x=0$ to $x=c$ where c is the period, suddenly becoming 0

and increasing to mc in the same way in the next period. Here, see Art. 133, q is $\frac{2\pi}{c}$,

$$mx = a_0 + a_1 \sin qx + b_1 \cos qx + \&c.$$
$$+ a_s \sin sqx + b_s \cos sqx + \&c.$$

a_0 is $\frac{1}{2}mc$,

$$a_s = \frac{2}{c}\int_0^c mx \,.\, \sin sqx \,.\, dx, \quad b_s = \frac{2}{c}\int_0^c mx \,.\, \cos sqx \,.\, dx.$$

Answer:

$$mx = \tfrac{1}{2}mc - \frac{mc}{\pi}(\sin qx + \tfrac{1}{2}\sin 2qx + \tfrac{1}{3}\sin 3qx + \tfrac{1}{4}\sin 4qx + \&c.).$$

2. Expand x in a series of sines and also in a series of cosines.

Answer: $x = 2(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \&c.)$ from $-\pi$ to π;

also $x = \frac{4}{\pi}(\sin x - \frac{1}{9}\sin 3x + \frac{1}{25}\sin 5x - \&c.)$ from 0 to $\frac{\pi}{2}$,

and $$x = \frac{\pi}{2} - \frac{4}{\pi}(\cos x + \tfrac{1}{9}\cos 3x + \tfrac{1}{25}\cos 5x + \&c.).$$

3. Prove $\frac{\pi}{4} = \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \&c.$

4. Show that

$$\epsilon^{ax} - \epsilon^{-ax} = \frac{2}{\pi}(\epsilon^{a\pi} - \epsilon^{-a\pi})\left\{\frac{\sin x}{1+a^2} - \frac{2\sin 2x}{2^2+a^2} + \frac{3\sin 3x}{3^2+a^2} - \&c.\right\}.$$

5. Integrate each of the above expansions.

230. 1. The radius of gyration of a sphere about a diameter being k and the radius a, prove that

$$k^2 = \tfrac{2}{5}a^2.$$

Here, since $x^2 + y^2 = a^2$, and the moment of inertia of a circular slice of radius y and thickness δx about its centre, is

$$\frac{\pi}{2}y^4 \,.\, \delta x.$$

The moment of inertia is

$$\int_0^a \pi m y^2 . dx \times y^2 = \int_0^a \pi m y^4 . dx = m\tfrac{8}{15}\pi a^5,$$

and the mass is $m\frac{4}{3}\pi a^3$.

2. In a paraboloid of height h and radius of base a, about the axis, $k^2 = \frac{1}{3}a^2$.

About the diameter of the base $k^2 = \frac{1}{6}(a^2 + h^2)$.

3. In a triangle of height h, about a line through the vertex parallel to the base, $k^2 = \frac{1}{2}h^2$.

About a line through centre of triangle parallel to base $k^2 = \frac{1}{18}h^2$.

231. Taylor's Theorem.

If a function of $x + h$, be differentiated with regard to x, h being supposed constant, we get the same answer as if we differentiate with regard to h, x being supposed constant.

This is evident. Call the function $f(u)$ where $u = x + h$. Then $\frac{d}{dx}f(u) = \frac{d}{du}f(u) \times \frac{du}{dx} = \frac{d}{du}f(u)$ as $\frac{du}{dx}$ is 1, and this is the same as $\frac{d}{dh}f(u)$ because

$$\frac{d}{dh}f(u) = \frac{d}{du}f(u) \times \frac{du}{dh} \text{ and } \frac{du}{dh} \text{ is } 1.$$

Assume that $f(x+h)$ may be expanded in a series of ascending powers of h.

$$f(x+h) = X_0 + X_1 h + X_2 h^2 + X_3 h^3 + \&c.\dots\dots\dots(1),$$

where X_0, X_1, X_2 &c. do not contain h.

$$\frac{df(x+h)}{dh} = 0 + X_1 + 2X_2 h + 3X_3 h^2 + \&c.\dots\dots(2),$$

$$\frac{df(x+h)}{dx} = \frac{dX_0}{dx} + \frac{dX_1}{dx} . h + \frac{dX_2}{dx} . h^2 + \&c.\dots\dots(3).$$

As (2) and (3) are identical

$$X_1 = \frac{dX_0}{dx}, \quad X_2 = \frac{1}{2}\frac{dX_1}{dx} = \frac{1}{1.2}\frac{d^2X_0}{dx^2},$$

$$X_3 = \frac{1}{3}\frac{dX_2}{dx} = \frac{1}{1.2.3}\frac{d^3X_0}{dx^3}.$$

Also if $h=0$ in (1) we find that $X_0=f(x)$. If we indicate $\frac{d^2}{dx^2}f(x)$ by $f''(x)$, then Taylor's Theorem is

$$\mathbf{f(x+h)=f(x)+hf'(x)+\frac{h^2}{1.2}f''(x)}$$
$$\mathbf{+\frac{h^3}{1.2.3}f'''(x)+\&c.} \quad \text{......(4).}$$

After having differentiated $f(x)$ twice, if we substitute 0 for x, let us call the result $f''(0)$; if we imagine 0 substituted for x in (4) we have

$$f(h)=f(0)+hf'(0)+\frac{h^2}{1.2}f''(0)$$
$$+\frac{h^3}{1.2.3}f'''(0)+\&c. \quad \text{............(5).}$$

Observe that we have no longer anything to do with the quantity which we call x. We may if we please use any other letter than h in (5); let us use the new letter x, and (5) becomes

$$f(x)=f(0)+xf'(0)+\frac{x^2}{1.2}f''(0)$$
$$+\frac{h^3}{1.2.3}f'''(0)+\&c. \quad \text{...........(6);}$$

which is called **Maclaurin's Theorem.**

The proof here given of Taylor's theorem is incomplete, as we have used an infinite series without proving it convergent. More exact proofs will be found in the regular treatises. Note that if x is time and $s=f(t)$ means distance of a body from some invariable plane in space; then if at the present time, which we shall call t_0, we know s and the velocity and the acceleration and $\frac{d^3s}{dt^3}$, &c.; that is, if we know all the circumstances of the motion absolutely correctly at the present time, then we can predict where the body will be at any future time, and we can say where the body was at any past time. It is a very far-reaching theorem and gives food for much speculation.

232. *Exercises on Taylor.* 1. Expand $(x+h)^n$ in powers of h.

Here $f(x)=x^n$, $f'(x)=nx^{n-1}$, $f''(x)=n(n-1)x^{n-2}$, &c. and hence

$$(x+h)^n = x^n + nhx^{n-1} + \frac{n(n-1)}{1.2}h^2x^{n-2} + \&c.$$

This is the Binomial Theorem, which is an example of Taylor.

2. Expand $\log(x+h)$ in powers of h.

Here $$f(x)=\log(x),\ f'(x)=\frac{1}{x},$$
$$f''(x)=-x^{-2},\ f'''(x)=+2x^{-3},$$

and hence $\log(x+h)=\log x + h\frac{1}{x} - \frac{h^2}{2}\frac{1}{x^2} + \frac{h^3}{3}\frac{1}{x^3} - \&c.$

If we put $x=1$ we have the useful formula

$$\log(1+h) = 0 + h - \frac{h^2}{2} + \frac{h^3}{3} - \&c.$$

3. Show that

$$\sin(x+h) = \sin x + h\cos x - \frac{h^2}{1.2}\sin x - \frac{h^3}{1.2.3}\cos x + \&c.$$

4. Show that

$$\cos(x+h) = \cos x - h\sin x - \frac{h^2}{1.2}\cos x + \frac{h^3}{1.2.3}\sin x + \&c.$$

5. What do 3 and 4 become when $x=0$?

233. *Exercises on Maclaurin.* 1. Expand $\sin x$ in powers of x.

$$\begin{aligned} f(x) &= \sin x, & f(0) &= 0, \\ f'(x) &= \cos x, & f'(0) &= 1. \\ f''(x) &= -\sin x, & f''(0) &= 0, \\ f'''(x) &= -\cos x, & f'''(0) &= -1, \\ f^{\text{iv}}(x) &= \sin x, & f^{\text{iv}}(0) &= 0, \\ &\&c. & f^{\text{v}}(0) &= 1. \end{aligned}$$

Hence $$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \&c.$$

2. Similarly $\cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \&c.$

Calculate from the above series the values of the sine and cosine of any angle, say 0·2 radians, and compare with what is given in books of mathematical tables.

3. Expand $\tan^{-1} x$. Another method is adopted.

The differential coefficient of $\tan^{-1} x$ is $\frac{1}{1+x^2}$, and by actual division this is $1 - x^2 + x^4 - x^6 + x^8 - \&c.$

Integrating this, term by term, we find

$$\tan^{-1} x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \tfrac{1}{9}x^9 - \&c.$$

We do not add a constant because when $x = 0$, $\tan^{-1} x = 0$.

4. Expand $\tan(1 - x)$ directly by Maclaurin.

5. Show that

$$a^x = 1 + x \log a + \frac{x^2}{\underline{|2}} (\log a)^2 + \frac{x^3}{\underline{|3}} (\log a)^3 + \&c.$$

6. Show that $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \&c.$

234. Expand $\epsilon^{i\theta}$, compare with the expansions of $\sin\theta$ and $\cos\theta$, and show that

$$\epsilon^{i\theta} = \cos\theta + i\sin\theta,$$

$$\epsilon^{-i\theta} = \cos\theta - i\sin\theta,$$

$$\cos\theta = \tfrac{1}{2}(\epsilon^{i\theta} + \epsilon^{-i\theta}),$$

$$\sin\theta = \frac{1}{2i}(\epsilon^{i\theta} - \epsilon^{-i\theta}).$$

Evidently $(\cos\theta \pm i\sin\theta)^n = \cos n\theta \pm i\sin n\theta$,

which is **Demoivre's Theorem.**

In solving cubic equations when there are three real roots, we find it necessary to extract the roots of unreal quantities by Demoivre. To find the qth root of $a + bi$ where a and b are given numerically. First write

$$a + bi = r(\cos\theta + i\sin\theta).$$

Then $r\cos\theta = a$, $r\sin\theta = b$, $r = \sqrt{a^2+b^2}$, $\tan\theta = \frac{a}{b}$. Calculate r and θ therefore.

Now the qth roots are, $r^{\frac{1}{q}}\left(\cos\frac{1}{q}\theta + i\sin\frac{1}{q}\theta\right)$,

$$r^{\frac{1}{q}}\left\{\cos\frac{1}{q}(2\pi+\theta) + i\sin\frac{1}{q}(2\pi+\theta)\right\},$$

$$r^{\frac{1}{q}}\left\{\cos\frac{1}{q}(4\pi+\theta) + i\sin\frac{1}{q}(4\pi+\theta)\right\} \text{ \&c.}$$

We easily see that there are only q, qth roots.

Exercise. Find the three cube roots of 8.

Write it $8(\cos 0 + i\sin 0)$, $8(\cos 2\pi + i\sin 2\pi)$, $8(\cos 4\pi + i\sin 4\pi)$ and proceed as directed.

235. The expansion of $\epsilon^{h\theta}$ is

$$1 + h\theta + \frac{1}{1.2}h^2\theta^2 + \frac{1}{1.2.3}h^3\theta^3 + \text{\&c.}$$

Now let θ stand for the operation $\frac{d}{dx}$, and we see that $f(x+h) = \epsilon^{h\theta}f(x)$; or $\epsilon^{h\frac{d}{dx}}f(x)$, symbolically represents Taylor's Theorem.

236. An equation which connects x, y and the differential coefficients is called a **Differential Equation.** We have already solved some of these equations.

The *order* is that of the highest differential coefficient.

The *degree* is the power of the highest differential coefficient. A differential equation is said to be *linear*, when it would be of the first degree, if y (the dependent variable), and all the differential coefficients, were regarded as unknown quantities. It will be found that if several solutions of a linear equation are obtained, their sum is also a solution.

Given any equation connecting x and y, containing constants; by differentiating one or more times we obtain sufficient equations to enable us to eliminate the constants.

Thus we produce a differential equation. Its primitive evidently contains n arbitrary constants if the equation is of the nth order.

Exercise. Eliminate a and b from

$$y = ax^2 + bx \text{......................} (1),$$

$$\frac{dy}{dx} = 2ax + b,\ \frac{d^2y}{dx^2} = 2a,\ b = \frac{dy}{dx} - x\frac{d^2y}{dx^2}.$$

Hence (1) becomes

$$y = \frac{x^2}{2}\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx} - x\frac{d^2y}{dx^2}\right),$$

or

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0 \text{ ....} (2).$$

If we solve (2) we find $y = Ax^2 + Bx$, where A and B are any arbitrary constants.

237. In the solution of Differential Equations we begin with equations of the **First Order** and the **First Degree.**

These are all of the type $M + N\frac{dy}{dx} = 0$, where M and N are functions of x and y. We usually write this in the shape

$$\mathbf{M \,.\, dx + N \,.\, dy = 0.}$$

Examples.

1. $(a + x)(b + y)\,dx + dy = 0$ or $(a + x)\,dx + \frac{1}{b + y}\,dy = 0.$

Integrating we have the general solution

$$ax + \tfrac{1}{2}x^2 + \log(b + y) = C,$$

where C is an arbitrary constant.

It is to be noticed here, as in any case when we can **separate the variables,** the solution is easy.

Thus if $f(x)F(y)\,.\,dx + \phi(x)\,.\,\psi(y)\,.\,dy = 0$ we have

$$\frac{f(x)\,.\,dx}{\phi(x)} + \frac{\psi(y)\,.\,dy}{F(y)} = 0,$$

and this can be at once integrated.

2. $$(1+x)\,y\,.\,dx+(1-y)\,x\,.\,dy=0,$$

or $$\left(\frac{1}{x}+1\right)dx+\left(\frac{1}{y}-1\right)dy=0.$$

Hence $$\log x+x+\log y-y=C,$$

or $$\log xy=C+y-x.$$

3. $$\frac{dx}{\sqrt{1-x^2}}+\frac{dy}{\sqrt{1-y^2}}=0.$$

Integrating, we have $\sin^{-1}x+\sin^{-1}y=c.$

This may be put in other shapes. Thus taking the sines of the two sides of the equation we have

$$x\sqrt{1-y^2}+y\sqrt{1-x^2}=C.$$

4. $\dfrac{dy}{dx}=\dfrac{x}{y}\cdot\dfrac{1+x}{1+y}$ becomes $(y+y^2)\,dy=(x+x^2)\,dx.$

Answer: $\frac{1}{2}y^2+\frac{1}{3}y^3=\frac{1}{2}x^2+\frac{1}{3}x^3+$ constant.

5. $\dfrac{xy\,(1+x^2)}{1+y^2}=\dfrac{dx}{dy}.$ Answer: $(1+x^2)(1+y^2)=cx^2.$

6. $\sin x\,.\cos y\,.\,dx-\cos x\,.\sin y\,.\,dy=0.$

Answer: $\cos y=c\cos x.$

7. $(y^2+xy^2)\,dx+(x^2-yx^2)\,dy=0.$

Answer: $\log\dfrac{x}{y}=c+\dfrac{y+x}{xy}.$

8. $\dfrac{y}{x}\dfrac{dy}{dx}+\sqrt{\dfrac{1+y^2}{1+x^2}}=0.$ Answer: $\sqrt{1+x^2}+\sqrt{1+y^2}=C.$

238. Sometimes we guess and find a **substitution** which answers our purpose. Thus to solve

$$\frac{dy}{dx}=\frac{y^2-x}{2xy},$$

we try $y=\sqrt{xv}$, and we find $\dfrac{dx}{x}+dv=0$, leading to

$$\log x+\frac{y^2}{x}=c.$$

Solve $$(y-x)\sqrt{1+x^2}\frac{dy}{dx}=n\,(1+y^2)^{\frac{3}{2}}.$$

239. If M and N are **homogeneous** functions of x and y of the same degree: assume $\mathbf{y=vx}$ and the equation reduces to the form of Art. 237.

Example 1. $ydx + (2\sqrt{xy} - x)\,dy = 0$. Assume $y = vx$,

$$dy = v \,.\, dx + x \,.\, dv,$$

$$vx \,.\, dx + (2x\sqrt{v} - x)(v \,.\, dx + x \,.\, dv) = 0,$$

$$(2xv^{\frac{3}{2}})\,dx + (2x^2\sqrt{v} - x^2)\,dv = 0,$$

$$\frac{2dx}{x} + \frac{2\sqrt{v} - 1}{v^{\frac{3}{2}}}\,dv = 0,$$

$$2\frac{dx}{x} + \left(\frac{2}{v} - \frac{1}{v^{\frac{3}{2}}}\right)dv = 0,$$

$$2\log x + 2\log v + 2v^{-\frac{1}{2}} = C,$$

$$\log xv + v^{-\frac{1}{2}} = C',$$

$$\log y + \sqrt{\frac{x}{y}} = C.$$

$$\text{Answer: } y = c\epsilon^{-\sqrt{\frac{x}{y}}},$$

where c is an arbitrary constant.

Example 2. $\dfrac{dy}{dx} = \sqrt{\dfrac{y}{x}} + 1 - \sqrt{\dfrac{x}{y}}$.

Let $y = vx$ and we find the answer

$$\frac{x^{\frac{1}{2}}}{x^{\frac{1}{2}} - y^{\frac{1}{2}}} + \log\{(x^{\frac{1}{2}} - y^{\frac{1}{2}})(x - y)^{\frac{1}{2}}\} = C.$$

Example 3. $\dfrac{dy}{dx} = \dfrac{y + \sqrt{x^2 + y^2}}{x}$.

$$\text{Answer: } x^2 = 2Ay + A^2.$$

Remember that two answers may really be the same although they may seem to be altogether different.

4. Solve $(x^3 + 3xy^2)\,dx + (y^3 + 3x^2y)\,dy = 0$.

Answer: $x^4 + 6x^2y^2 + y^4 = C$.

5. Solve $$y^2 + (xy + x^2)\frac{dy}{dx} = 0.$$

Answer: $3xy + x^2 + 2y^2 = Cx^{\frac{1}{2}}(2y + x)^{\frac{3}{2}}$.

6. Solve $\left(x - y\cos\frac{y}{x}\right)dx + x\cos\frac{y}{x}\,.\,dy = 0.$

Answer: $x = c\epsilon^{-\sin\frac{y}{x}}$.

7. $(y - x)\,dy + y\,.\,dx = 0.$ Answer: $2y = c\epsilon^{-\frac{x}{y}}$.

8. $xdy - y\,.\,dx - \sqrt{x^2 + y^2}\,.\,dx = 0.$ Answer: $x^2 = c^2 + 2cy$.

9. $x + y\dfrac{dy}{dx} = 2y.$ Answer: $(x - y)\,\epsilon^{\frac{x}{x-y}} = C$.

240. Of the form $(ax + by + c)\,dx + (a'x + b'y + c')\,dy = 0.$

Assume $\mathbf{x = w + \alpha}$, $\mathbf{y = v + \beta}$, and choose α and β so that the constant terms disappear.

Thus if $(3x - 2y + 4)\,dx + (2x - y + 1)\,dy = 0$; as $dx = dw$ and $dy = dv$, we have

$$(3w + 3\alpha - 2v - 2\beta + 4)\,dw + (2w + 2\alpha - v - \beta + 1)\,dv = 0.$$

Now choose α and β so that

$$3\alpha - 2\beta + 4 = 0 \text{ and } 2\alpha - \beta + 1 = 0,$$

or $$-\alpha + 2 = 0, \text{ or } \alpha = 2,\ \beta = 5.$$

Therefore the substitution ought to be $x = w + 2$, $y = v + 5$, and the equation becomes a homogeneous one.

Exercise. $(3y - 7x + 7)\,dx + (7y - 3x + 3)\,dy = 0.$ $\alpha = 1$ $\beta = 0$

Answer: $(y - x + 1)^2(y + x - 1)^5 = c$.

Exercise. $$\frac{dy}{dx} + \frac{2x - y + 1}{2y - x - 1} = 0.$$

Answer: $x^2 - xy + y^2 + x - y = c$.

241. Exact Differential Equations are those which have been derived by the differentiation of a function of x and y, not being afterwards multiplied or divided by any function of x and y. Consult Art. 83.

$Mdx + Ndy = 0$ is an exact differential equation if

$$\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right) \text{ because } M = \frac{df(x, y)}{dx}, \quad N = \frac{d}{dy} f(x, y),$$

the primitive being $f(x, y) = c$. It will be found that

$$(x^3 - 3x^2y)\, dx + (y^3 - x^3)\, dy = 0,$$

is an exact differential equation.

Then $$x^3 - 3x^2y = \frac{df(x, y)}{dx}.$$

Integrating therefore, as if y were constant, and adding Y an unknown function of y, instead of a constant,

$$f(x, y) = \tfrac{1}{4}x^4 - x^3y + Y.$$

Differentiating as if x were constant, and equating to N, we have

$$-x^3 + \frac{dY}{dy} = y^3 - x^3,$$

$$\frac{dY}{dy} = y^3 \text{ and hence } Y = \tfrac{1}{4}y^4 + c.$$

Hence $$\tfrac{1}{4}x^4 - x^3y + \tfrac{1}{4}y^4 + c = 0,$$

where c is any arbitrary constant.

242. Any equation $M \,.\, dx + N \,.\, dy = 0$ may be made exact by multiplying by some function of x called an **Integrating Factor.** See Art. 83. For the finding of such factors, students are referred to the standard works on differential equations.

243. Linear equations of the first order.

These are of the type

$$\mathbf{\frac{dy}{dx} + Py = Q} \quad \text{.....................(1)},$$

where P and Q are functions of x.

The general solution is this. Let $\int P \,.\, dx$ be called X, then

$$\mathbf{y = \epsilon^{-X} \left\{ \int \epsilon^{X} \,.\, Q \,.\, dx + C \right\}} \text{..............(2)},$$

where C is an arbitrary constant.

No proof of this need be given, other than that if the value of y is tried, it will be found to satisfy the equation. Here is the trial :—

$$(2) \text{ is the same as } y\epsilon^X = \int \epsilon^X Q dx + C \quad \text{.........(3).}$$

Differentiating, and recollecting that $\frac{dX}{dx} = P$, (3) becomes

$$\frac{dy}{dx}\epsilon^X + y\epsilon^X P = \epsilon^X Q \quad \text{..................(4),}$$

or $\frac{dy}{dx} + Py = Q$, the original equation.

To obtain the answer (2) from (1), multiply (1) by ϵ^X and we get (4); integrate (4) and we have (3); divide by ϵ^X and we have (2).

We have, before, put (1) in the form

$$(\theta + P)\,y = Q \text{ or } y = (\theta + P)^{-1}\,Q,$$

and now we see the general meaning of the inverse operation $(\theta + P)^{-1}$. In fact if $\int P\,.\,dx$ be called X,

$$\mathbf{(\theta + P)^{-1}\,Q \text{ means, } \epsilon^{-X}\left\{\int \epsilon^X\,.\,Q\,.\,dx + C\right\}} \quad \text{...(5).}$$

Thus if Q is 0, $(\theta + P)^{-1}\,0 = C\epsilon^{-X}$. Again, if P is a constant a, and if Q is 0, then $(\theta + a)^{-1}\,0 = C\epsilon^{-ax}$, where C is an arbitrary constant. We had this in Art. 168.

Again, if Q is also a constant, say n,

$$(\theta + a)^{-1}\,n = \epsilon^{-ax}\left\{\int n\epsilon^{ax}\,.\,dx + C\right\}$$

$$= C\epsilon^{-ax} + \frac{n}{a} \quad \text{.........................(6),}$$

where C is an arbitrary constant. See Art. 169.

Again, if $Q = \epsilon^{bx}$,

$$(\theta + a)^{-1}\,\epsilon^{bx} = \epsilon^{-ax}\left\{\int \epsilon^{(a+b)\,x}\,.\,dx + C\right\}$$

$$= C\epsilon^{-ax} + \frac{1}{a+b}\epsilon^{bx} \quad \text{..................(7).}$$

It is easy to show that when $a = -b$, $y = (C + x)\,\epsilon^{-ax}$.

If $Q = b \sin(cx + e)$,

$$(\theta + a)^{-1} b \sin(cx + e) = \epsilon^{-ax}\left\{b \int \epsilon^{ax} \sin(cx + e)\,.\,dx + C\right\}$$

$$= C\epsilon^{-ax} + \frac{b}{\sqrt{a^2 + c^2}} \sin\left(cx + e - \tan^{-1}\frac{c}{a}\right) \quad \text{.......(8).}$$

244. *Example.* In an electric circuit let the voltage at the time t be V, and let C be the current, the resistance being R and the self-induction L. We have the well-known equation

$$V = RC + L\frac{dC}{dt},$$

or

$$\frac{dC}{dt} + \frac{R}{L}C = \frac{1}{L}V.$$

Now

$$\int \frac{R}{L}\,.\,dt = \frac{R}{L}t,$$

and hence $$C = \epsilon^{-\frac{R}{L}t}\left\{\frac{1}{L}\int \epsilon^{\frac{R}{L}t}\,V\,.\,dt + \text{constant } A\right\} \quad \text{......(1).}$$

Of this we may have many cases.

1st. Let V at time 0, suddenly change from having been a constant V_1, to another constant V_2. Put $V = V_2$ therefore in the above answer, and we have

$$C = \epsilon^{-\frac{R}{L}t}\left\{\frac{1}{R}V_2\epsilon^{\frac{R}{L}t} + A\right\}$$

$$= \frac{V_2}{R} + A\epsilon^{-\frac{R}{L}t}.$$

To determine A we know that $C = \frac{V_1}{R}$ when $t=0$; therefore $\frac{V_1}{R} = \frac{V_2}{R} + A$ so that $A = \frac{V_1 - V_2}{R}$ and hence

$$C = \frac{V_2}{R} - \frac{V_2 - V_1}{R}\epsilon^{-\frac{R}{L}t} \quad \text{..................(2).}$$

Thus if V_1 was 0, $C = \frac{V_2}{R}\left(1 - \epsilon^{-\frac{R}{L}t}\right)$ ....(3),

showing how a current rises when a circuit is *closed.*

Again if V_2 is 0, $C = \frac{V_1}{R}\epsilon^{-\frac{R}{L}t}$ ....(4),

showing how a current falls when an electromotive force is destroyed.

Students ought to plot these values of C with time.

Take as an example, $V_2 = 100$ volts in (3), $R = 1$ ohm, $L = \cdot 01$ Henry.

Again take $V_1 = 100$ volts in (4). Compare Art. 169.

2nd. Let V at time 0 suddenly become

$$V_0 \sin qt,$$

$$C = \epsilon^{-\frac{R}{L}t}\left\{\frac{V_0}{L}\int \epsilon^{\frac{R}{L}t} . \sin qt . dt + A\right\},$$

$$C = \epsilon^{-\frac{R}{L}t}\left\{\frac{V_0}{L}\frac{\epsilon^{\frac{R}{L}t}\left(\frac{R}{L}\sin qt - q\cos qt\right)}{\frac{R^2}{L^2} + q^2} + A\right\}.$$

This becomes $C = A\epsilon^{-\frac{R}{L}t} + \frac{V_0}{\sqrt{R^2 + L^2q^2}}\sin(qt - e)$...(5),

where $$\tan e = \frac{Lq}{R}.$$

The constant A, of the evanescent term $A\epsilon^{-\frac{R}{L}t}$, depends upon the initial conditions; thus if $C = 0$ when $t = 0$,

$$0 = A - \frac{V_0}{\sqrt{R^2 + L^2q^2}}\sin e,$$

$$\therefore 0 = A - \frac{V_0}{\sqrt{R^2 + L^2q^2}} \cdot \frac{Lq}{\sqrt{R^2 + L^2q^2}},$$

or $$A = V_0Lq/(R^2 + L^2q^2).$$

Students ought to plot curves of several examples, taking other initial conditions.

245. *Example.* A body of mass M, moving with velocity v, in a fluid which exerts a resistance to its motion, of the amount fv, is acted upon by a force whose amount is F at the time t. The equation is

$$M\frac{dv}{dt}+fv=F.$$

Notice that this is exactly the electrical case, if M stands for L, f for R, F for Volts V, v for C; and we have exactly the same solutions if we take it that F is constant, or that F alters from one constant value F_1 to another F_2, or that F follows a law like $F_0 \sin qt$.

This analogy might be made much use of by lecturers on electricity. A mechanical model to illustrate how electric currents are created or destroyed could easily be made.

The solutions of Linear Differential Equations with constant coefficients have such practical uses in engineering calculations that we took up the subject and gave many examples in Chap. II. Possibly the student may do well now to read Art. 151 over again at this place.

246. *Example.* $x\dfrac{dy}{dx}=ay+x+1$,

or
$$\frac{dy}{dx}-\frac{a}{x}y=1+\frac{1}{x},$$

$$\int\frac{-a}{x}.dx=X=-a\log x.$$

Observe that $\epsilon^{-a\log x}=x^{-a}$, $\epsilon^{a\log x}=x^{a}$.

Hence
$$y=x^a\left\{\int x^{-a}\left(1+\frac{1}{x}\right)dx+C\right\},$$

$$y=x^a\left\{\int(x^{-a}+x^{-a-1})\,dx+C\right\},$$

$$y=x^a\left\{\frac{x^{1-a}}{1-a}+\frac{x^{-a}}{-a}+C\right\},$$

$$y=\frac{x}{1-a}-\frac{1}{a}+Cx^a,$$

the answer, where C is an arbitrary constant.

247. There being continuous lubricating liquid between the surfaces AB and EF as of a brass and a journal. $OC=h_0$ the nearest distance between them. At the distance x, measured along the arc OA, let the thickness be h. Anywhere in the normal line there, representing the thickness, let there be a point in the liquid at the distance y from the journal, and let the velocity of the liquid there be u. Then if p be the pressure it can be shown that

$$\frac{d^2u}{dy^2}=\frac{1}{\mu}\frac{dp}{dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

if μ is the coefficient of viscosity of the lubricant, and u_0 the linear velocity of the journal, and u is the velocity of the liquid at any place; we have no space for the reasoning from (1)

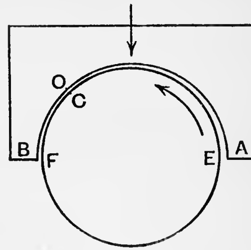

Fig. 103.

leading to
$$\frac{d^2p}{dx^2}+\frac{3}{h}\cdot\frac{dh}{dx}\frac{dp}{dx}+\frac{6\mu u_0}{h^3}\frac{dh}{dx}=0 \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Let
$$\frac{dp}{dx}=\phi,$$

then
$$\frac{d\phi}{dx}+\frac{3}{h}\cdot\frac{dh}{dx}\cdot\phi+\frac{6\mu u_0}{h^3}\frac{dh}{dx}=0.$$

This is of the shape (1) Art. 243, h in terms of x being given.

Let $X=\int P\,.\,dx=\int\frac{3}{h}\cdot\frac{dh}{dx}\,.\,dx=3\log h$, $\epsilon^X=h^3$. Hence

$$\phi=h^{-3}\left\{\int -h^3\frac{6\mu u_0}{h^3}\frac{dh}{dx}dx+C\right\},\ -\phi=-\frac{dp}{dx}=h^{-3}(6\mu u_0 h+C)=\frac{6\mu u_0}{h^2}+\frac{C}{h^3}.$$

The solution depends upon the law of variation of h. The real case is most simply approximated to by $h=h_0+ax^2$: using this we find

$$p=C'-\frac{6\mu u_0}{2h_0}\left(\frac{x}{h}+\frac{1}{\sqrt{ah_0}}\tan^{-1}x\sqrt{\frac{a}{h_0}}\right)$$
$$-C\left\{\frac{x}{h^2}+\frac{3}{2h_0}\left(\frac{x}{h}+\frac{1}{\sqrt{ah_0}}\tan^{-1}x\sqrt{\frac{a}{h_0}}\right)\right\},$$

$$p=C'-C\frac{x}{h^2}-\left(\frac{x}{h}+\frac{1}{\sqrt{ah_0}}\tan^{-1}x\sqrt{\frac{a}{h_0}}\right)\left(\frac{6\mu u_0}{2h_0}+\frac{3C}{2h_0}\right).$$

If students were to spend a few weeks on this example they might be induced to consult the original paper by Prof. O. Reynolds in the *Phil. Trans.* vol. 177, in which he first explained to engineers the theory of lubrication.†

Numerical Exercise. Let $OB=2{\cdot}59$, $OA=11{\cdot}09$ centimetres.

$\mu=2{\cdot}16$, h_0 or $OC=0{\cdot}001135$, $a={\cdot}0000082$, $u_0=80$ cm. per second.

Calculate C and C', assuming $p=0$ at B and at A.

Now calculate the pressure for various values of x and graph it on squared paper. The friction per square cm. being $\mu \frac{du}{dy}$ at $y=0$, the total friction F will be found to be

$$-F=\int \frac{h}{2} \frac{dp}{dx} dx + \mu u_0 \int \frac{dx}{h}$$

between the limits A and B. The total load on the bearing is $\int p \, . \, dx$ between the limits, if AB covers only a small part of the journal, and may be calculated easily in any case.

The bearing is supposed to be infinitely long at right angles to the paper in fig. 103, but forces are reckoned per cm. of length.

248. *Example.* Solve

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 2\epsilon^{3x}.$$

Writing this in the form $(\theta^2 - 4\theta + 3)\, y = 2\epsilon^{3x}$,

$$y = (\theta^2 - 4\theta + 3)^{-1} 2\epsilon^{3x}.$$

Now $$(\theta^2 - 4\theta + 3)^{-1} = \tfrac{1}{2}\left(\frac{1}{\theta - 3} - \frac{1}{\theta - 1}\right).$$

Indeed we need not have been so careful about the $\frac{1}{2}$ as it is obvious that the general solution is the sum of the two $(\theta - 3)^{-1}$ and $(\theta - 1)^{-1}$, each multiplied by an arbitrary constant.

Anyhow, $$y = \frac{\epsilon^{3x}}{\theta - 3} - \frac{\epsilon^{3x}}{\theta - 1},$$

and this by (7) Art. 243 is

$$(C + x)\, \epsilon^{3x} - \{C'\epsilon^{x} + \tfrac{1}{2}\epsilon^{3x}\},$$

or $$y = (C_1 + x)\, \epsilon^{3x} + C_2\epsilon^{x}.$$

249. Equations like $\frac{dy}{dx} + Py = Qy^n$, where P and Q, as before, are functions of x only.

Divide all across by y^n and substitute $z = y^{1-n}$, and the equation becomes linear.

Example. $$(1 - x^2)\frac{dy}{dx} - xy = axy^2.$$

Substituting $z = y^{-1}$ we find

$$\frac{dz}{dx} + \frac{xz}{1-x^2} = -\frac{ax}{1-x^2},$$

$$\int \frac{x}{1-x^2}\,dx = -\tfrac{1}{2}\log(1-x^2),$$

$$\epsilon^{-\frac{1}{2}\log(1-x^2)} = (1-x^2)^{-\frac{1}{2}},$$

$$z = (1-x^2)^{\frac{1}{2}}\left\{\int (1-x^2)^{-\frac{3}{2}}(-ax)\,dx + C\right\}.$$

Answer:

$$y^{-1} = (1-x^2)^{\frac{1}{2}}\{-a(1-x^2)^{-\frac{1}{2}} + C\} = -a + C\sqrt{1-x^2}.$$

Exercise. $$x\frac{dy}{dx} + y = y^2 \log x.$$

Answer: $\dfrac{1}{y} = 1 + Cx + \log x.$

250. 1. Given $\left(\dfrac{dy}{dx}\right)^2 - a^2y^2 = 0.$

This is an equation of the **first order** and **second degree.**

Solve for $\dfrac{dy}{dx}$ and we find two results,

$$\frac{dy}{dx} - ay = 0, \text{ so that } \log y - ax - A_1 = 0,$$

$$\frac{dy}{dx} + ay = 0, \text{ so that } \log y + ax - A_2 = 0.$$

Hence the solution is

$$(\log y - ax - A_1)(\log y + ax - A_2) = 0.$$

It will be found that each value of y only involves *one* arbitrary constant, although two are shown in the equation.

2. Given $$1 + \left(\frac{dy}{dx}\right)^3 = x.$$

This is an equation of the **first order** and **third degree.**

Hence $$\frac{dy}{dx} = (x-1)^{\frac{1}{3}},$$

and $$y = \tfrac{3}{4}(x-1)^{\frac{4}{3}} + C.$$

3. Given $$\left(\frac{dy}{dx}\right)^2 - 7\frac{dy}{dx} + 12 = 0.$$

This is an equation of the **first order** and **second degree.**

$$\left(\frac{dy}{dx} - 4\right)\left(\frac{dy}{dx} - 3\right) = 0.$$

$$(y - 4x + c_1)(y - 3x + c_2) = 0.$$

251. Clairaut's equation is of the first order and of any degree

$$y = xp + f(p) \text{ ....(1)},$$

where p is $\frac{dy}{dx}$ and $f(p)$ is any function of $\frac{dy}{dx}$.

Differentiate with regard to x, and we find

$$\left\{x + \frac{d}{dp}f(p)\right\}\frac{dp}{dx} = 0 \text{ ....(2)}.$$

So, either $$\frac{dp}{dx} = 0 \text{ ....(3)}$$

or $$x + \frac{d}{dp}f(p) = 0 \text{ ....(4)}$$

will satisfy the equation.

If $$\frac{dp}{dx} = 0,\ p = c.$$

Substituting this in (1) we have

$$y = cx + f(c) \text{ ....(5)},$$

which is the complete solution.

Eliminating p now between (1) and (4) we obtain another solution which contains no arbitrary constant. Much may be said about this **Singular Solution** as it is called. It is the result of eliminating c from the family of curves (5), and is, therefore, their **Envelope.** See Art. 224.

Example of Clairaut's equation.

$$y = xp + \frac{m}{p}.$$

We have the general solution (5) ... $y = cx + \frac{m}{c}$, a family

of straight lines the members of which differ in the values of their c.

(4) is $x = -\frac{d}{dp}\left(\frac{m}{p}\right)$ or $x = +\frac{m}{p^2}$ or $p = \sqrt{\frac{m}{x}}$.

Hence $y = 2\sqrt{mx}$ or $y^2 = 4mx$, a parabola which we found to be the envelope of the family in Art. 224.

This curve satisfies the original equation, because in any infinitesimal length, the values of x, y and $\frac{dy}{dx}$ are the same for it as for a member of the family of straight lines.

252. If a differential equation is of the form

$$\frac{d^n y}{dx^n} = f(x),$$

it can be at once solved by **successive integration.** We have had many examples of this in our work already.

253. Equations of the form $\frac{d^2y}{dx^2} = f(y)$; multiply by $2\frac{dy}{dx}$ and integrate and we have

$$\left(\frac{dy}{dx}\right)^2 = 2\int f(y) \, . \, dy + C.$$

Extracting the square root, the equation may be solved, as the variables are separated.

Thus let $$\frac{d^2y}{dx^2} = a^2 y.$$

Proceeding as above,

$$\left(\frac{dy}{dx}\right)^2 = 2\int a^2 y \, . \, dy + C = a^2y^2 + C,$$

$$\frac{dy}{dx} = \sqrt{a^2y^2 + C},$$

$$\frac{dy}{\sqrt{a^2y^2 + C}} = dx.$$

Integrating we find

$$x = \frac{1}{a}\log\left\{ay + \sqrt{a^2y^2 + C}\right\} + C' \text{ ....(1).}$$

If this equation (1) is put in the shape

$$c\epsilon^{ax} = ay + \sqrt{a^2y^2 + C},$$

it becomes $$c^2\epsilon^{2ax} - 2ayc\epsilon^{ax} = C,$$

$$y = \frac{c}{2a}\epsilon^{ax} - C'\epsilon^{-ax},$$

or $$y = A\epsilon^{ax} + B\epsilon^{-ax} \quad\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

which looks different from (1) but is really the same. (2) is what we obtain at once if we solve according to the rule for linear equations, Art. 159.

254. Solve $\dfrac{d^3y}{dx^3} = a\dfrac{d^2y}{dx^2}$, an equation of the third order and first degree.

Let $$\frac{d^2y}{dx^2} = q, \text{ then } \frac{dq}{dx} = aq,$$

$$q = b\epsilon^{ax} \text{ so that } \frac{dy}{dx} = \frac{b}{a}\epsilon^{ax} + C,$$

$$y = \frac{b}{a^2}\epsilon^{ax} + Cx + C',$$

or $y = A\epsilon^{ax} + Cx + C'$, where A, C, C' are arbitrary constants. This also might have been solved by the rule for linear equations, Art. 159.

255. Solve $$a\frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}},$$

$$a\frac{dp}{dx} = (1+p^2)^{\frac{1}{2}}, \text{ if } p = \frac{dy}{dx},$$

$$\frac{a\,.\,dp}{\sqrt{1+p^2}} = dx \text{ so that } \frac{x}{a} = \log\{p + \sqrt{p^2+1}\} + C',$$

or $$C\epsilon^{\frac{x}{a}} - p = \sqrt{p^2+1}.$$

Squaring, we find $p = \frac{1}{2}C\epsilon^{\frac{x}{a}} - \dfrac{1}{2C}\epsilon^{-\frac{x}{a}} = \dfrac{dy}{dx}$.

Integrating this we have

$$y=\frac{a}{2}C\epsilon^{\frac{x}{a}}+\frac{a}{2C}\epsilon^{-\frac{x}{a}}+c,$$

where C and c are arbitrary constants.

256. General Exercises on Differential Equations.

(1) $(a^2+y^2)\,dx+\sqrt{a^2-x^2}\,.\,dy=0.$

Answer: $\sin^{-1}\frac{x}{a}+\frac{1}{a^2}\tan^{-1}\frac{y}{a}=c.$

(2) $\frac{dy}{dx}+\frac{x}{1+x^2}y=-\frac{1}{2x(1+x^2)}.$

Answer: $y=\frac{1}{\sqrt{1+x^2}}\left\{c-\frac{1}{2}\log\frac{1+\sqrt{1+x^2}}{x}\right\}.$

(3) $\frac{dy}{dx}+y\cos x=\frac{\sin 2x}{2}.$

Answer: $y=\sin x-1+C\epsilon^{-\sin x}.$

(4) $\frac{dy}{dx}\left(\frac{dy}{dx}+y\right)=x(x+y).$

Answer: $(2y-x^2-c)\{\log(x+y-1)+x-c\}=0.$

(5) $\frac{dy}{dx}+2xy=2ax^3y^3.$

Answer: $y=\{C\epsilon^{2x^2}+\frac{1}{2}a(2x^2+1)\}^{-\frac{1}{2}}.$

(6) $1=\left(\frac{dy}{dx}\right)^2+\frac{2x}{y}\frac{dy}{dx}.$ Answer: $y^2=2cx+c^2.$

(7) $x\frac{dy}{dx}+y=y^2\log x.$ Answer: $y=(cx+\log x+1)^{-1}.$

(8) $y=2x\frac{dy}{dx}+y^2\left(\frac{dy}{dx}\right)^3.$ Answer: $y^2=2cx+c^3.$

(9) Solve $\left(1+\frac{y^2}{x^2}\right)dx-2\frac{y}{x}dy=0.$

1st, after the manner of the Exercise of Art. 241.

2nd, as a homogeneous equation.

Answer: $x^2-y^2=cx.$

(10) Solve $\frac{2x \, . \, dx}{y^3} + \left(\frac{1}{y^2} - \frac{3x^2}{y^4}\right) dy = 0$ after the manner of Art. 241. Answer: $x^2 - y^2 = cy^3$.

(11) $\frac{dy}{dx} - \frac{2y}{x+1} = (x+1)^3$.

Answer: $y = (x+1)^2 \{\frac{1}{2}(x+1)^2 + C\}$.

(12) $(x+y)^2 \frac{dy}{dx} = a^2$. Answer: $y - a \tan^{-1} \frac{x+y}{a} = c$.

(13) $xy(1 + xy^2) \frac{dy}{dx} = 1$. Answer: $\frac{1}{x} = 2 - y^2 + c\epsilon^{-\frac{1}{2}y^2}$.

(14) $\left(\frac{dy}{dx}\right)^2 - \frac{a^2}{x^2} = 0$.

Answer: $(y - a \log x - c)(y + a \log x - c) = 0$.

(15) $\frac{d^4y}{dx^4} + 4\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 0$.

Answer: $y = (a_0 + a_1 x + a_2 x^2 + a_3 x^3) e^{-x}$.

(16) $\frac{d^2x}{dt^2} - 2f\frac{dx}{dt} + f^2 x = \epsilon^t$.

Answer: $x = (a_1 + a_2 t)\epsilon^{ft} + \frac{\epsilon^t}{(f-1)^2}$.

(17) $\frac{dy}{dx} - \frac{ny}{x+1} = \epsilon^x (x+1)^n$.

Answer: $y = (x+1)^n (\epsilon^x + c)$.

(18) $\frac{d\theta}{d\phi} = \sin(\phi - \theta)$. Answer: $\cot\left\{\frac{\pi}{4} - \frac{\phi - \theta}{2}\right\} = \phi + c$.

(19) $(1 - x^2)\frac{dy}{dx} - xy = axy^2$. Answer: $\frac{1}{y} = c\sqrt{1 - x^2} - a$.

(20) $\frac{d^3y}{dx^3} - 2\frac{dy}{dx} + 4y = \epsilon^x \cos x$.

Answer: $y = C_1 \epsilon^{-2x} + \epsilon^x \left\{\left(C_2 - \frac{x}{20}\right)\cos x + \left(C_3 + \frac{3x}{20}\right)\sin x\right\}$.

(21) Change the independent variable from x to t in $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$, if $x = \sin t$, and solve the equation.

(22) Also in $(a^2+x^2)\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}=0$, if $x=a\tan t$, and solve the equation.

257. 1. Prove $\dfrac{d^2s}{dt^2}=-\dfrac{d^2t}{ds^2}\Big/\left(\dfrac{dt}{ds}\right)^3$,

and 2. $\dfrac{d^3s}{dt^3}=-\left\{\dfrac{dt}{ds}\cdot\dfrac{d^3t}{ds^3}-3\left(\dfrac{d^2t}{ds^2}\right)^2\right\}\div\left(\dfrac{dt}{ds}\right)^5$.

3. Prove as $\dfrac{dy}{dx}=\dfrac{dy}{dt}\Big/\dfrac{dx}{dt}$, so we have also

$$\frac{d^2y}{dx^2}=\left(\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{d^2x}{dt^2}\cdot\frac{dy}{dt}\right)\div\left(\frac{dx}{dt}\right)^3,$$

and find the equivalent expression for $\dfrac{d^3y}{dx^3}$.

4. If $x=\epsilon^t$, show that as $x\dfrac{dy}{dx}=x\dfrac{dy}{dt}\div\dfrac{dx}{dt}$ this equals $\dfrac{dy}{dt}$.

Also $$x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dt^2}-\frac{dy}{dt}=\left(\frac{d}{dt}-1\right)\frac{dy}{dt},$$

and $$x^3\frac{d^3y}{dx^3}=\left(\frac{d}{dt}-2\right)\left(\frac{d}{dt}-1\right)\frac{dy}{dt}.$$

5. Change the independent variable from x to t if $x=\epsilon^t$, in $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+n^2y=0$, and solve the equation.

258. If we try to find by the method of Art. 47 the length of the arc of an ellipse, we encounter the second class Elliptic Integral which is called **E (k, x)**. It may be evaluated in an infinite series. Its value has been calculated for values of **k** and **x** and tabulated in Mathematical tables.

When the angle through which a pendulum swings is not small, and we try to find the periodic time, we encounter the first class Elliptic Integral which is called **F (k, x)**. It can be shown that the integral of any algebraic expression involving the square root of a polynomial of the third or fourth degree may be made to depend on one or more of the three integrals

$$F(k,\,x)=\int_0^x\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},\ \text{or}\ F(k,\,\theta)=\int_0^\theta\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}};$$

$$E(k, x)=\int_0^x \sqrt{\frac{1-k^2x^2}{1-x^2}} \, . \, dx, \text{ or } E(k, \theta)=\int_0^\theta \sqrt{1-k^2\sin^2\theta} \, . \, d\theta \, ;$$

$$\pi(n, k, x)=\int_0^x \frac{dx}{(1+nx^2)\sqrt{(1-x^2)(1-k^2x^2)}}, \text{ or}$$

$$\pi(n, k, \theta)=\int_0^\theta \frac{d\theta}{(1+n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}.$$

k, which is always positive and less than 1, is called the *modulus*. **n**, which is any real number, is called the *parameter*.

The change from the x form to the θ form is effected by the substitution, $x=\sin\theta$. When the limits of F and E are 1 and 0 in the x case, or $\frac{\pi}{2}$ and 0 in the θ case, the integrals are called *complete*, and the letters K and E merely are used for them. θ is called the *amplitude* and $\sqrt{1-k^2\sin^2\theta}$ is called by the name $\Delta\theta$.

If $u=F(k, x)=F(k, \theta)$, then in dealing with functions which have the same k if we use the names

$$\theta=\text{am}\, u,$$

$$x=\text{sn}\, u \text{ (in words, } x \text{ is the sine of the amplitude of } u),$$

$$\sqrt{1-x^2}=\text{cn}\, u \text{ (or } \sqrt{1-x^2} \text{ is the cosine of the amplitude of } u),$$

$$\sqrt{1-k^2x^2}=\text{dn}\, u \text{ (or } \sqrt{1-k^2x^2} \text{ is the delta of the amplitude of } u),$$

it is found that

$$\text{sn}^2 u+\text{cn}^2 u=1, \quad \text{dn}^2 u+k^2 . \text{sn}^2 u=1, \quad \frac{d}{du}(\text{am}\, u)=\text{dn}\, u, \text{ \&c.}$$

Also $$\text{am}(-u)=-\text{am} . u, \text{ \&c.}$$

Also $$\text{sn}(u \pm v)=\frac{\text{sn}\, u . \text{cn}\, v . \text{dn}\, v \pm \text{cn}\, u . \text{sn}\, v . \text{dn} . u}{1-k^2\text{sn}^2 u . \text{sn}^2 v},$$

and similar relations for $\text{cn}(u \pm v)$ and $\text{dn}(u \pm v)$.

Expressions for $\text{sn}(u+v)+\text{sn}(u-v)$, &c. follow. Also for

$$\text{sn}\, 2u, \quad \text{cn}\, 2u, \quad \text{dn}\, 2u.$$

So that there is as complete a set of formulae connecting these elliptic functions, as connect the Trigonometrical functions, and there are series by means of which tables of them may be calculated. Legendre published tables of the first and second class integrals, and as they have known relations with those of the third class, special values of these and of the various elliptic functions may be worked out. If complete tables of them existed, it is possible that these functions might be familiar to practical men.

259. To return to our **differentiation of functions of two or more variables.**

1. If $u = z^2 + y^3 + zy$ and $z = \sin x$ and $y = \epsilon^x$, then $\left(\frac{du}{dy}\right) = 3y^2 + z$, $\left(\frac{du}{dz}\right) = 2z + y$, $\frac{dy}{dx} = \epsilon^x$, $\frac{dz}{dx} = \cos x$, and hence $\frac{du}{dx} = (3y^2 + z)\,\epsilon^x + (2z + y)\cos x$. If this is expressed all in terms of x we have the same answer that we should have had, if we had substituted for y and z in terms of x in u originally, and differentiated directly.

2. If $u = \sqrt{\frac{v^2 - w^2}{v^2 + w^2}}$, where v and w are functions of x, find $\frac{du}{dx}$.

3. If $\sin(xy) = mx$, find $\frac{dy}{dx}$.

4. If $u = \sin^{-1}\frac{z}{y}$, where z and y are functions of x, find $\frac{dy}{dx}$.

5. If $u = \tan^{-1}\frac{z}{y}$, show that $du = \frac{y\,.\,dz - z\,.\,dy}{y^2 + z^2}$.

260. *Exercise.* Try if the equation

$$\frac{d^2v}{dx^2} = \frac{1}{\kappa}\frac{dv}{dt} \quad\text{.......................(1)}$$

has a solution like $v = \epsilon^{\alpha x}\sin(qt + \gamma x)$, and if so, find α and γ, and make it fit the case in which $v = 0$ when $x = \infty$, and $v = a\sin qt$ where $x = 0$. We leave out the brackets of $\left(\frac{dv}{dt}\right)$ &c.

$$\frac{dv}{dx} = \alpha\epsilon^{\alpha x}\sin(qt + \gamma x) + \epsilon^{\alpha x}\gamma\cos(qt + \gamma x),$$

$$\frac{d^2v}{dx^2} = \alpha^2\epsilon^{\alpha x}\sin(qt + \gamma x) + \alpha\gamma\epsilon^{\alpha x}\cos(qt + \gamma x)$$
$$+ \alpha\gamma\epsilon^{\alpha x}\cos(qt + \gamma x) - \epsilon^{\alpha x}\gamma^2\sin(qt + \gamma x).$$

Also $$\frac{dv}{dt} = q\epsilon^{\alpha x}\cos(qt + \gamma x),$$

so that to satisfy (1) for all values of t and x

$$\alpha^2-\gamma^2=0 \text{ or } \alpha=\pm\gamma,$$

and $$\alpha\gamma+\alpha\gamma=\frac{q}{\kappa}.$$

As $\frac{q}{\kappa}$ is not zero, $\alpha=+\gamma$ only,

$$2\alpha^2=\frac{q}{\kappa},\ \alpha=\pm\sqrt{\frac{q}{2\kappa}}=\gamma.$$

Hence we have

$$v=A\epsilon^{\alpha x}\sin(qt+\alpha x)+B\epsilon^{-\alpha x}\sin(qt-\alpha x),$$

where A and B are any constants, and $\alpha=\sqrt{\frac{2\pi n}{2\kappa}}$ or $\sqrt{\frac{\pi n}{\kappa}}$ if $q=2\pi n$. Now if $v=0$ when $x=\infty$, obviously $A=0$. If $v=a\sin qt$ where $x=0$, obviously $B=a$.

Hence the answer is

$$v=a\epsilon^{-x\sqrt{\frac{\pi n}{\kappa}}}\sin\left(2\pi nt-x\sqrt{\frac{\pi n}{\kappa}}\right)\ \ldots\ldots(2).$$

261. Let a **point** P be **moving in a curved path** SPQ; let $AP=x$, $BP=y$,

$\frac{dx}{dt}$ and $\frac{dy}{dt}$ being the velocities in the directions OX and OY,

$\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$ being the accelerations in the directions OX and OY.

Let $OP=r$, $BOP=\theta$, $x=r\cos\theta$, $y=r\sin\theta$. The acceleration or velocity of P in any direction is to be obtained just as we resolve forces. Thus the velocity in the direction r is

$$\frac{dx}{dt}\cos\theta+\frac{dy}{dt}\sin\theta\ldots(1),$$

and in the direction PT which is at right angles to r, the velocity is

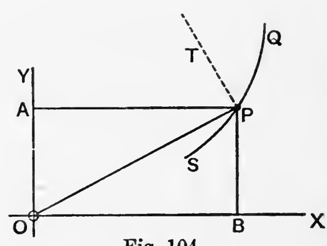

Fig. 104.

$$-\frac{dx}{dt}\sin\theta+\frac{dy}{dt}\cos\theta\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Now differentiating x and y, as functions of the variables r and θ, since $\left(\frac{dx}{dr}\right) = \cos\theta$ and $\left(\frac{dx}{d\theta}\right) = -r\sin\theta$,

$$\frac{dx}{dt} = \frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt} \quad \text{..............(3)},$$

$$\frac{dy}{dt} = \frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt} \quad \text{..............(4)}.$$

Solving (3) and (4) for $\frac{dr}{dt}$ and $r\frac{d\theta}{dt}$, we find

$$\frac{dr}{dt} = \frac{dx}{dt}\cos\theta + \frac{dy}{dt}\sin\theta \quad \text{..............(5)},$$

$$r\frac{d\theta}{dt} = -\frac{dx}{dt}\sin\theta + \frac{dy}{dt}\cos\theta \quad \text{..............(6)}.$$

From (1) and (2) we see therefore that $\frac{dr}{dt}$ is the velocity in the direction OP and that $r\frac{d\theta}{dt}$ is the velocity in the direction PT. Some readers may think this obvious.

Now if we resolve the x and y accelerations in the direction of OP and PT, as we did the velocities, and if we again differentiate (3) and (4) with regard to t, we find

$$\text{Acceleration in direction } OP = \frac{d^2x}{dt^2}\cos\theta + \frac{d^2y}{dt^2}\sin\theta \quad \text{...(7)},$$

$$\text{Acceleration in direction } PT = -\frac{d^2x}{dt^2}\sin\theta + \frac{d^2y}{dt^2}\cos\theta \quad \text{...(8)}.$$

And

$$\frac{d^2x}{dt^2} = \left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\}\cos\theta - \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\sin\theta \quad \text{...(9)},$$

$$\frac{d^2y}{dt^2} = \left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\}\sin\theta + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\cos\theta \quad \text{...(10)}.$$

And hence, the acceleration in the direction r is (and this is not very obvious without our proof),

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \quad \text{....................(11)},$$

and the acceleration in the direction PT is

$$2\frac{dr}{dt}\frac{d\theta}{dt}+r\frac{d^2\theta}{dt^2}, \text{ or } \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) \qquad \text{.........(12).}$$

$r^2\frac{d\theta}{dt}$ is usually called h. It is evidently twice the area per second swept over by the radius vector, and (12) is $\frac{1}{r}\frac{dh}{dt}$.

262. If the force causing motion is a **central Force,** an attraction in the direction PO, which is a function of r per unit mass of P, say $f(r)$; or $mf(r)$ on the mass m at P; then (12) is 0, or $r^2\frac{d\theta}{dt}=$ constant, or h constant. Hence under the influence of a central force, the radius vector sweeps out equal areas in equal times.

Equating $mf(r)$ to the mass multiplied by the acceleration in the direction PO we have

$$f(r)=r\left(\frac{d\theta}{dt}\right)^2-\frac{d^2r}{dt^2} \qquad \text{.................(13).}$$

But $r^2\frac{d\theta}{dt}=h$ a constant. As r is a function of θ

$$\frac{dr}{dt}=\frac{dr}{d\theta}\cdot\frac{d\theta}{dt}=\frac{dr}{d\theta}\frac{h}{r^2},$$

$$\frac{d^2r}{dt^2}=\left\{\frac{d^2r}{d\theta^2}\frac{h}{r^2}-2\frac{h}{r^3}\left(\frac{dr}{d\theta}\right)^2\right\}\frac{h}{r^2},$$

and we can use these values in (13) to eliminate t.

If we use $\frac{1}{u}$ for r, (13) simplifies into

$$\mathbf{f(r)=h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right)} \qquad \text{..............(14).}$$

If $f(r)=ar^{-n}$ or au^n, an attraction varying inversely as the nth power of the distance, $\frac{d^2u}{d\theta^2}+u=\frac{a}{h^2}u^{n-2}=bu^{n-2}$, say.

Multiplying by $2\frac{du}{d\theta}$ and integrating, we have

$$\left(\frac{du}{d\theta}\right)^2+u^2=\frac{2b}{n-1}u^{n-1}+c \qquad \text{...........(15).}$$

Thus let the law be that of the **inverse square,** $f(r) = ar^{-2}$ or au^2; (14) becomes

$$au^2 = h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right),$$

$$\frac{d^2u}{d\theta^2} + u = \frac{a}{h^2} = b, \text{ say.}$$

Let $w = u - b$, then

$$\frac{d^2w}{d\theta^2} + w = 0.$$

The solution of this is,

$$w = A\cos(\theta + B),$$

and it may be written

$$u = \frac{1}{r} = \frac{a}{h^2}\{1 + e\cos(\theta - \alpha)\} \quad \text{.........(16).}$$

This is known as the polar equation to a conic section, the focus being the pole. The nature of the conic section depends upon the initial conditions.

(15) enables us, when **given the shape of path,** to find the law of central force which produces it. Thus if a particle describes an ellipse under an attraction always directed towards the centre, it will be found that the force of attraction is proportional to distance. It is easier when given this law to find the path. For if the force is proportional to PO, the x component of it is proportional to x, and the y component to y. If the accelerations in these directions are written down, we find that simple harmonic motions of the same period are executed in these two directions and the composition of such motions is well known to give an elliptic path. If the law of attraction is **the inverse cube** or $f(r) = ar^{-3} = au^3$, (14) becomes $\dfrac{d^2u}{d\theta^2} + u = \dfrac{a}{h^2}u$.

If $\qquad \dfrac{a}{h^2} - 1 = \alpha^2, \quad u = A\epsilon^{\alpha\theta} + B\epsilon^{-\alpha\theta}.$

If $\qquad 1 - \dfrac{a}{h^2} = \beta^2, \quad u = A\sin\beta\theta + B\cos\beta\theta,$

giving curiously different answers according to the initial conditions of the motion.

263. If $x = r\cos\theta$, $y = r\sin\theta$, so that if y is a function of x, r must be a function of θ; if u is any function of x and y, it is also a function of r and θ.

Express $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$ in terms of the polar co-ordinates r and θ.

$$\left(\frac{du}{dr}\right) = \left(\frac{du}{dx}\right)\frac{dx}{dr} + \left(\frac{du}{dy}\right)\frac{dy}{dr} \quad\text{..............(1)},$$

θ being supposed constant, and

$$\left(\frac{du}{d\theta}\right) = \left(\frac{du}{dx}\right)\frac{dx}{d\theta} + \left(\frac{du}{dy}\right)\frac{dy}{d\theta} \quad\text{..............(2)},$$

r being supposed constant.

Now $\quad \frac{dx}{d\theta}$ if r is constant $= -r\sin\theta$,

$\frac{dy}{d\theta}$ if r is constant $= r\cos\theta$,

$\frac{dx}{dr}$ if θ is constant $= \cos\theta$,

$\frac{dy}{dr}$ if θ is constant $= \sin\theta$.

Treating $\left(\frac{du}{dx}\right)$ and $\left(\frac{du}{dy}\right)$ in (1) and (2) as unknown, and finding them, we have

$$\left(\frac{du}{dx}\right) = \cos\theta \cdot \left(\frac{du}{dr}\right) - \frac{1}{r}\sin\theta \cdot \left(\frac{du}{d\theta}\right) \quad\text{.........(3)},$$

$$\left(\frac{du}{dy}\right) = \sin\theta \cdot \left(\frac{du}{dr}\right) + \frac{1}{r}\cos\theta \cdot \left(\frac{du}{d\theta}\right) \quad\text{.........(4)}.$$

Notice that in $\left(\frac{du}{dx}\right)$, the bracket means that y is supposed to be constant in the differentiation.

In $\left(\frac{du}{dr}\right)$, it is θ that is supposed to be constant.

In (3) or (4) treat $\left(\frac{du}{dx}\right)$ or $\left(\frac{du}{dy}\right)$ as u is treated, and find $\frac{d^2u}{dx^2}$ and $\frac{d^2u}{dy^2}$. However carefully one works, mistakes are likely to occur, and this practice is excellent as one must think very carefully at every step. Prove that

$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}=\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}+\frac{1}{r^2}\frac{d^2u}{d\theta^2}\text{.............(5).}$$

264. Sometimes instead of x, y and z, we use r, θ, ϕ co-ordinates for a point in space. Imagine that from the centre of the earth O (Fig. 105), we have OZ the axis of the earth, OX a line at right angles to OZ, the plane ZOX being through Greenwich; OY a line at right angles to the other two. The position of a point P is defined by x its distance

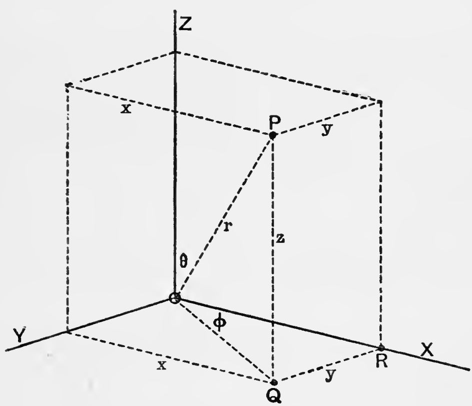

Fig. 105.

from the plane ZOY, y its distance from the plane ZOX, z its distance from the equatorial plane YOX. Let r be OP the distance of the point from O. Let ϕ be the west longitude or the angle between the planes POZ and XOZ; or if Q be the foot of the perpendicular from P upon XOY, the angle QOX is ϕ. Let θ be the co-latitude or the angle POZ. Then it is easy for anyone who has done practical geometry to see that, drawing the lines in the figure, QRO is a

right angle and $OR=x$, $QR=y$, also PQO is a right angle, $x=r\sin\theta\,.\cos\phi$, $y=r\sin\theta\,.\sin\phi$, $z=r\cos\theta$. If u is a given function of x, y and z, it can be expressed in terms of r, θ and ϕ, by making substitutions. It is an excellent exercise to prove

$$\frac{du}{dx}=\sin\theta\cos\phi\,\frac{du}{dr}+\frac{\cos\theta\,.\cos\phi}{r}\frac{du}{d\theta}-\frac{\sin\phi}{r\sin\theta}\frac{du}{d\phi},$$

$$\frac{du}{dy}=\sin\theta\sin\phi\,\frac{du}{dr}+\frac{\cos\theta\,.\sin\phi}{r}\frac{du}{d\theta}+\frac{\cos\phi}{r\sin\theta}\cdot\frac{du}{d\phi},$$

$$\frac{du}{dz}=\cos\theta\,\frac{du}{dr}-\frac{\sin\theta}{r}\frac{du}{d\theta}.$$

It will be noticed that we easily slip into the habit of leaving out the brackets indicating partial differentiation.

The average student will not have the patience, possibly he may not be able to work sufficiently accurately, to prove that

$$\mathbf{\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}=\frac{d^2u}{dr^2}+\frac{1}{r^2}\frac{d^2u}{d\theta^2}}$$
$$\mathbf{+\frac{1}{r^2\sin^2\theta}\cdot\frac{d^2u}{d\phi^2}+\frac{2}{r}\frac{du}{dr}+\frac{\cot\theta}{r^2}\frac{du}{d\theta}}\text{......(A).}$$

This relation is of very great practical importance.

265. The foundation of much practical work consists in understanding the equation

$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}=\frac{1}{\kappa}\frac{du}{dt}\quad\text{..............(1),}$$

where t is time. For example, we must solve (1) in Heat Conduction Problems if u is temperature, or in case $\frac{du}{dt}=0$ and u is electric or magnetic potential, or velocity potential, in Hydrodynamics.

(1) is usually written

$$\mathbf{\nabla^2u=\frac{1}{\kappa}\frac{du}{dt}}\text{.......................(2).}$$

We see then in (A) the form that $\nabla^2 u$ takes, in terms of r, θ and ϕ co-ordinates.

We know that if u is symmetrical about the axis of z, that is, if u is independent of ϕ, the above expression becomes

$$\nabla^2 u = \frac{d^2u}{dr^2} + \frac{1}{r^2}\frac{d^2u}{d\theta^2} + \frac{2}{r}\frac{du}{dr} + \frac{\cot\theta}{r^2}\frac{du}{d\theta} \quad \text{.........(3).}$$

266. Students are asked to work out every step of the following long example with great care. The more time taken, the better. This example contains all the essential part of the theory of **Zonal Spherical Harmonics,** so very useful in Practical Problems in Heat, Magnetism, Electricity, Hydrodynamics and Gravitation. When u is independent of ϕ we sometimes write (2) in the form

$$\frac{du}{dt} = \frac{\kappa}{r^2}\left\{\frac{d}{dr}\left(r^2\frac{du}{dr}\right) + \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{du}{d\theta}\right)\right\} \quad \text{......(1),}$$

u being a function of time t, r and θ. The student had better see if it is correct according to (3).

If $\frac{du}{dt} = 0$, show that the equation becomes

$$r^2\frac{d^2u}{dr^2} + 2r\frac{du}{dr} + \cot\theta\frac{du}{d\theta} + \frac{d^2u}{d\theta^2} = 0 \quad \text{.........(2).}$$

Try if there is a solution of the form $u = RP$ where R is a function of r only, and P is a function of θ only, and show that we have

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{2r}{R}\frac{dR}{dr} = -\cot\theta\frac{1}{P}\frac{dP}{d\theta} - \frac{1}{P}\frac{d^2P}{d\theta^2} \quad \text{......(3).}$$

Now the left-hand side contains only r and no θ, the right-hand side contains only θ and no r. Consequently each of them must be a constant. Let this constant be called C and we have

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{RC}{r^2} = 0 \quad \text{.................(4),}$$

$$\frac{d^2P}{d\theta^2} + \cot\theta\frac{dP}{d\theta} + PC = 0 \text{..............(5).}$$

There is no restriction as to the value of C, and it must be the same in (4) and (5), and then the product of the two answers is a value of u which will satisfy (2). The solutions of many linear Partial Differential Equations are obtained in the form of a product in this way. There are numberless other solutions but we can make good practical use of these.

We have then reduced our solution of the Partial Differential Equation (1), to the solution of a pair of ordinary differential equations (4) and (5). Now a solution of (4) may be found by trial to be r^m, and when this is the case we have a method (see Art. 268) of proving the general solution to be

$$R = Ar^m + Br^{-(m+1)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

where C is $m(m+1)$; anyhow (6) will be found by trial to answer. Using this way of writing C in (5) and letting $\cos\theta = \mu$, we find that we have an equation called Legendre's Equation, an ordinary linear equation of the 2nd order

$$\frac{\mathbf{d}}{\mathbf{d}\mu}\left\{(\mathbf{1}-\mu^2)\frac{\mathbf{dP}}{\mathbf{d}\mu}\right\} + \mathbf{m}(\mathbf{m}+\mathbf{1})\mathbf{P} = \mathbf{0} \quad \ldots\ldots(7).$$

We now find it convenient to restrict m. Let m be a positive integer, and try if there is a solution of (7) in the form

$$P = 1 + A_1\mu + A_2\mu^2 + A_3\mu^3 + \&c.$$

Calling it $P_m(\mu)$ or $P_m(\theta)$, the answers are found to be

$P_0(\theta) = 1$, if m is put 0, $\qquad P_1(\theta) = \mu$, if m is put 1,

$P_2(\theta) = \frac{3}{2}\mu^2 - \frac{1}{2}$, if m is put 2, $P_3(\theta) = \frac{5}{2}\mu^3 - \frac{3}{2}\mu$, if m is put 3,

$P_4(\theta) = \frac{35}{8}\mu^4 - \frac{30}{8}\mu^2 + 3$, if m is put 4.

A student will find it a good exercise to work out these to P_9. My pupils have worked out tables of values of P_0, P_1, P_2, &c., to P_7 for every degree from $\theta = 0$ to $\theta = 180°$. See the *Proceedings of the Physical Society*, London, Nov. 14, 1890, where clear instructions are given as to the use of Zonal Harmonics in solving practical problems.

We see then that

$$\left(\mathbf{Ar^m} + \frac{\mathbf{B}}{\mathbf{r^{m+1}}}\right)\mathbf{P_m}(\theta) \quad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

is a solution of (1). A practical problem usually consists in this:—Find u to satisfy (1) and also to satisfy certain limiting conditions. In a great number of cases terms like (8) have only to be added together to give the complete solution wanted.

In the present book I think that it would be unwise to do more in this subject than to set the above very beautiful exercise as an example of easy differentiation.

(8) is usually called **The Solid Zonal Harmonic of the mth degree, $\mathbf{P_m(\theta)}$ is called the Surface Zonal Harmonic of the mth degree.**

267. In many axial problems, u is a function only of time and of r the distance of a point from an axis, and we require solutions of (1) which in this case becomes

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} = \frac{1}{\kappa}\frac{du}{dt} \quad \text{..................(1).}$$

Let us, as before, look for a solution in the form

$$u = RT \quad \text{..........................(2),}$$

where R is a function of r only and T is a function of t only. (1) becomes

$$T\frac{d^2R}{dr^2} + \frac{1}{r}T\frac{dR}{dr} = \frac{1}{\kappa}R\frac{dT}{dt}.$$

Dividing by RT

$$\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{r}\frac{1}{R}\frac{dR}{dr} = \frac{1}{\kappa}\frac{1}{T}\frac{dT}{dt} = -\mu^2, \text{ say,}$$

where μ^2 is a constant.

Then $\frac{dT}{T} = -\kappa\mu^2 dt$ or $\log T = -\kappa\mu^2 t + c$, or

$$T = C\epsilon^{-\kappa\mu^2 t} \quad \text{........................(3),}$$

where C is an arbitrary constant. We must now solve

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \mu^2 R = 0 \quad \text{..................(4).}$$

Let $r = \frac{x}{\mu}$ and (4) becomes

$$\mathbf{\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + R = 0} \quad \text{..................(5).}$$

Assume now that there is a solution of (5) of the shape

$$R = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + Fx^6 + Gx^7 + \&c.,$$

we find that $A = C = E = G = 0$ and in fact that

$$R = \mathbf{1} - \frac{\mathbf{x}^2}{\mathbf{2}^2} + \frac{\mathbf{x}^4}{\mathbf{2}^2\mathbf{4}^2} - \frac{\mathbf{x}^6}{\mathbf{2}^2\mathbf{4}^2\mathbf{6}^2} + \frac{\mathbf{x}^8}{\mathbf{2}^2\mathbf{4}^2\mathbf{6}^2\mathbf{8}^2} - \mathbf{\&c.} \quad \ldots\ldots(6).$$

This is an important series first used by Fourier, although it has Bessel's name. It is called the **Zeroth Bessel** and the symbol $J_0(x)$ is used for it. Tables are published which enable us for any value of x to find $J_0(x)$. Thus then $R = J_0(\mu r)$ is a solution of (4), and hence

$$u = C\epsilon^{-\kappa\mu^2 t} J_0(\mu r) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

is a solution of (1). Any solution of (1) needed in a practical problem is usually built up of the sum of terms like (7), where different values of μ and different values of C are selected to suit the given conditions.

268. In the linear differential equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

when P and Q are functions of x, **if we know a particular solution,** say $y = v$, **we can find the general solution.**

Substitute $y = vu$, and we get

$$v\frac{d^2u}{dx^2} + 2\left(\frac{dv}{dx} + Pv\right)\frac{du}{dx} = 0 \quad \ldots\ldots\ldots\ldots(2).$$

Calling $\frac{du}{dx} = u'$, (2) becomes

$$v\frac{du'}{dx} + \left(2\frac{dv}{dx} + Pv\right)u' = 0,$$

or

$$\frac{du'}{u'} + 2\frac{dv}{v} + P\,.\,dx = 0,$$

or

$$\log u' + \log v^2 + \int P\,.\,dx = \text{constant}.$$

Let $\int P \,.\, dx = X$ then u' or $\frac{du}{dx} = A \frac{1}{v^2} \epsilon^{-X}$,

$$u = B + A \int \frac{1}{v^2} \epsilon^{-X} \,.\, dx \quad \text{.................(3).}$$

Thus we find the general solution

$$y = Bv + Av \int \frac{1}{v^2} \epsilon^{-X} \,.\, dx \quad \text{...............(4),}$$

where A and B are arbitrary constants. Even if the right-hand side is not zero, the above substitution will enable the solution to be found, if v is a solution when the right-hand side is 0.

Easy Example. One solution of

$$\frac{d^2y}{dx^2} + a^2x = 0 \text{ is } y = \cos ax.$$

Find the general solution.

Here $P = 0$ so that $\int P \,.\, dx = X = 0$.

Hence $$y = B \cos ax + A \cos ax \int \frac{dx}{\cos^2 ax};$$

and as $$\int \frac{dx}{\cos^2 ax} = \frac{1}{a} \tan ax,$$

we have as the general solution

$$y = B \cos ax + C \sin ax.$$

Exercise. We find by trial that $y = x^m$ is a solution of $x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - m(m+1)\, y = 0$, see Art. 266.

Show that the general solution is $y = Ax^m + \frac{B}{x^{m+1}}$.

Exercise. We find by trial that $y = \epsilon^{ax}$ is a solution of $\frac{d^2y}{dx^2} = a^2y$, show that the general solution is $y = A\epsilon^{ax} + B\epsilon^{-ax}$.

Exercise. We saw that $u = P_m(\mu)$ is a solution of Legendre's equation Art. 266, prove that $u = AP_m(\mu) + BQ_m(\mu)$ is the general solution, where

$$Q_m(\mu) = P_m(\mu) \int \frac{d\mu}{(1-\mu^2)\{P_m(\mu)\}^2},$$

$Q_m(\mu)$ or $Q_m(\theta)$ is called the **Surface Zonal Harmonic of the second kind.**

Exercise. We saw that $J_0(x)$ was a solution of the Bessel equation (5), show that the general solution is $AJ_0(x) + BK_0(x)$, where

$$K_0(x) = \int \frac{dx}{x\{J_0(x)\}^2},$$

K_0 (x) is called the Zeroth Bessel of the second kind.

269. Conduction of Heat. If material supposed to be homogeneous has a plane face AB. If at the point P which is at the distance x from AB, the temperature is v, and we imagine the temperature the same at all points in the same plane as P parallel to AB (that is, we are only considering flow of heat at right angles to the plane AB), and if $\frac{dv}{dx}$ is the rate of rise of temperature per centimetre at P, then $-k\frac{dv}{dx}$ is the amount of heat flowing per second through a square centimetre of area like PQ, in the direction of increasing x. This is the definition of k, the conductivity. We shall imagine k constant.

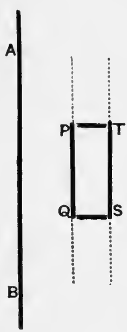

Fig. 106.

k is the heat that flows per second per square centimetre, when the temperature gradient is 1. Let us imagine PQ exactly a square centimetre in area. Now what is the flow across TS, or what is the value of $-k\frac{dv}{dx}$ at the new place, which is $x + \delta x$ from the plane AB? Observe that $-k\frac{dv}{dx}$ is a function of x; call it $f(x)$ for a moment, then the

space $PQTS$ receives heat $f(x)$ per second, and gives out heat $f(x+\delta x)$ per second.

Now $$f(x+\delta x)-f(x)=\delta x\frac{df(x)}{dx}.$$

These expressions are of course absolutely true only when δx is supposed to be smaller and smaller without limit.

We have then come to the conclusion that $-\delta x\frac{d}{dx}f(x)$ is being *added* to the space $PQTS$ every second: this is

$$-\delta x\frac{d}{dx}\left(-k.\frac{dv}{dx}\right) \text{ or } +k.\delta x\frac{d^2v}{dx^2}.$$

But the volume is $1\times\delta x$, and if w is the weight per cubic centimetre, and if s is the specific heat or the heat required to raise unit weight one degree in temperature, then if t is time in seconds,

$$w.\delta x.s\frac{dv}{dt}$$

also measures the rate per second at which the space receives heat. Hence

$$k.\delta x.\frac{d^2v}{dx^2}=w.\delta x.s.\frac{dv}{dt},$$

or $$\frac{d^2v}{dx^2}=\frac{ws}{k}.\frac{dv}{dt}\text{.......................(1).}$$

This is the **fundamental equation** in conduction of heat problems. Weeks of study would not be thrown away upon it. It is in exactly this same way that we arrive at the fundamental equations in Electricity and Hydrodynamics.

If flow were not confined to one direction we should have the equation (1) of Art. 265. $\frac{k}{ws}$ is often called the **diffusivity** for heat of a material, and is indicated by the Greek letter κ; ws is the capacity for heat of unit volume of the material.

Let us write (1) as

$$\frac{d^2v}{dx^2}=\frac{1}{\kappa}\frac{dv}{dt}\text{ ....(2).}$$

270. It will be found that there are innumerable solutions of this equation, but there is only one that suits a particular problem. Let us imagine the average temperature everywhere to be 0 (it is of no consequence what zero of temperature is taken, as only differences enter into our calculations), and that

$$V = a \sin 2\pi nt, \text{ or } a \sin qt \quad \ldots\ldots\ldots\ldots\ldots(3),$$

is the law according to which the temperature changes at the skin where x is 0; n or $\frac{q}{2\pi}$ means the number of complete periodic changes per second. Now we have carefully examined the cycle of temperature of steam in the clearance space of a steam-cylinder, and it follows sufficiently closely a simple harmonic law for us to take this as a basis of calculation. Take any periodic law one pleases, it consists of terms like this, and any complicated case is easily studied. Considering the great complexity of the phenomena occurring in a steam-cylinder, we think that this idea of simple harmonic variation at the surface of the metal, is a good enough hypothesis for our guidance. Now we take it that although the range of temperature of the actual skin of the metal is much less than that of the steam, it is probably roughly proportional to it, so we take a to be proportional to the **range of temperature** of the steam. We are not now considering the **water in the cylinder,** on the skin and in pockets, as requiring itself to be heated and cooled; this heating and cooling occurs with enormous rapidity, and the less there is of such water the better, so it ought to be drained away rapidly. But besides this function of the water, the layer on the skin acts as creating in the actual metal, a range of temperature which approaches that in the steam itself, keeping a large. Our n means the number of revolutions of the engine per second.

To suit this problem we find the value of v everywhere and at all times to be what is given in (2) of Art. 260,

$$v = a\epsilon^{-x\sqrt{\frac{\pi n}{\kappa}}} \sin\left(2\pi nt - x\sqrt{\frac{\pi n}{\kappa}}\right) \quad \ldots\ldots\ldots(4).$$

This is the answer for an infinite mass of material with one plane face. It is approximately true in the wall of a thick

cylinder, if the outside is at temperature 0. If the outside is at temperature v' and the thickness of the metal is b we have only to add a term $\frac{v'}{b}x$ to the expression (4). This shows the effect of a **steam-jacket** as far as mere conductivity is concerned. The steam-jacket diminishes the value of a also. Taking (4) as it stands, the result ought to be very carefully studied. At any point at the depth x there is a simple harmonic rise and fall of temperature every revolution of the engine; but the range gets less and less as the depth is greater and greater. Note also that the changes lag more as we go deeper. This is exactly the sort of thing observed in the buried thermometers at Craigleith Quarry, Edinburgh. The changes in temperature were 1st of 24 hours period, 2nd of 1 year period; we give the yearly periodic changes, the average results of eighteen years' observations.

Depth in feet below surface	Yearly range of temperature Fahrenheit	Time of highest temperature
3 feet	16·138	August 14
6 feet	12·296	„ 26
12 feet	8·432	Sept. 17
24 feet	3·672	Nov. 7

Observations at 24 feet below the surface at Calton Hill, Edinburgh, showed highest temperature on January 6th.

Now let us from (4) find the rate at which heat is flowing through a square centimetre; that is, calculate $-k\frac{dv}{dx}$ for any instant; calling $\sqrt{\frac{\pi n}{\kappa}} = \alpha$,

$$\frac{dv}{dx} = -\alpha a \epsilon^{-\alpha x} \sin(2\pi nt - \alpha x) - \alpha a \epsilon^{-\alpha x} \cos(2\pi nt - \alpha x),$$

where $x = 0$, that is, at the skin, it becomes when multiplied

by $-k$; $\left(-k\dfrac{dv}{dx}\right)_{x=0} = +ka\alpha\,\{\sin 2\pi nt + \cos 2\pi nt\}$

$$= ka\alpha\sqrt{2}\sin\left(2\pi nt + \frac{\pi}{4}\right), \text{ by Art. 116.}$$

This is + for half a revolution when heat is flowing *into* the metal, and it is − for the other half revolution when heat is flowing out of the metal. Let us find **how much flows in**; it will be equal to the amount flowing out. It is really the same as

$$ka\alpha\sqrt{2}\int_0^{\frac{1}{2}\tau}\sin 2\pi nt\,.\,dt = ka\alpha\sqrt{2}\,.\,\frac{\tau}{\pi}, \text{ where } \tau = \frac{1}{n},$$

$$= a\sqrt{\frac{2kws}{n\pi}}.$$

That is, it is inversely proportional to the square root of the speed and is proportional to the range of temperature.

We have here a certain simple exact mathematical result; students must see in what way it can be applied in an engineering problem when the phenomena are very complicated. We may take it as furnishing us with a roughly correct notion of what happens. That is, we may take it that the latent heat lost by steam in one operation is less with steam jacketing, and with drying of the skin; is proportional to the range of temperature of the steam, to the surface exposed at cut off, and inversely proportional to the square root of the speed. Probably what would diminish it more than anything else, would be the admixture with the steam of some air, or an injection of flaming gas, or some vapour less readily condensed than steam. When we use many terms of a Fourier development instead of merely one, we are led to the result that the heat lost in a steam cylinder in one stroke is

$$(\theta_1 - \theta_3)\left(b + \frac{c}{r}\right)A/\sqrt{n},$$

where θ_1 is the initial temperature and θ_3 the back pressure temperature, r the ratio of cut off, n the number of revolutions per minute, A the area of the piston b and c constants

whose values depend upon the type of engine and the arrangements as to drainage and jacketing.

271. Students will find it convenient to keep by them a good **list of integrals.** It is most important that they collect such a list for their own use, but we have always found that it gets mislaid unless bound up in some book of reference. We therefore print such a list here. Repetition was unavoidable.

Fundamental cases:

1. $\int x^m \,.\, dx = \frac{1}{m+1}\, x^{m+1}.$

2. $\int \frac{1}{x} \,.\, dx = \log x.$

3. $\int \epsilon^x \,.\, dx = \epsilon^x.$

4. $\int a^x \,.\, dx = \frac{1}{\log a}\, a^x.$

5. $\int \cos mx \,.\, dx = \frac{1}{m} \sin mx.$

6. $\int \sin mx \,.\, dx = -\frac{1}{m} \cos mx.$

7. $\int \cot x \,.\, dx = \log (\sin x).$

8. $\int \tan x \,.\, dx = -\log (\cos x).$

9. $\int \tan x \,.\, \sec x \,.\, dx = \sec x.$

10. $\int \sec^2 x \,.\, dx = \tan x.$

11. $\int \operatorname{cosec}^2 x \,.\, dx = -\cot x.$

12. $\int \frac{dx}{\cos^2 ax} = \frac{1}{a} \tan ax.$

13. $\int \frac{dx}{\sin^2 ax} = -\frac{1}{a} \cot ax.$

14. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}.$

15. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}.$

16. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{x}{a}.$

17*. $\int \cosh ax \,.\, dx = \frac{1}{a} \sinh ax.$

* From 17 to 23 we have used the symbols (called Hyperbolic sines, cosines &c.)

$$\sinh x = \tfrac{1}{2}(\epsilon^x - \epsilon^{-x}), \text{ and } \operatorname{cosech} x = \frac{1}{\sinh x},$$

$$\cosh x = \tfrac{1}{2}(\epsilon^x + \epsilon^{-x}), \operatorname{sech} x = \frac{1}{\cosh x},$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\epsilon^{2x} - 1}{\epsilon^{2x} + 1}, \coth x = \frac{1}{\tanh x}.$$

Also if $y = \sinh x$, $x = \sinh^{-1} y$.

It is easy to prove that

$$\sinh(a+b) = \sinh a \,.\, \cosh b + \cosh a \,.\, \sinh b,$$
$$\cosh(a+b) = \cosh a \cosh b + \sinh a \,.\, \sinh b,$$
$$\sinh(a-b) = \sinh a \cosh b - \cosh a \sinh b,$$
$$\cosh(a-b) = \cosh a \cosh b - \sinh a \sinh b,$$
$$\sinh 2a = 2 \sinh a \,.\, \cosh a,$$
$$\cosh 2a = \cosh^2 a + \sinh^2 a = 2\cosh^2 a - 1,$$
$$= 2\sinh^2 a + 1.$$

If we assume that $\sqrt{-1}$, or i as we call it, submits to algebraic rules and $\iota^2 = -1$, $\iota^3 = -i$, $\iota^4 = 1$, $\iota^5 = i$ &c. we can write $a + bi$ as $r(\cos\theta + i\sin\theta)$, where $r^2 = a^2 + b^2$, and $\tan\theta = \frac{b}{a}$. It is easy to extract the nth root of $a + bi$: being $r^{1/n}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$, and by adding on 2π to θ as many times as we please, we get n, nth roots.

We also find that $\epsilon^{ia} = \cos a + i\sin a$,

$$\epsilon^{-ia} = \cos a - i\sin a.$$

If $z = a + bi = r(\cos\theta + i\sin\theta) = r \,.\, \epsilon^{i\theta}$,

$$\log z = \log r + i\theta = \tfrac{1}{2}\log(a^2 + b^2) + i\tan^{-1}\frac{b}{a}.$$

18. $\int \sinh ax \, . \, dx = \frac{1}{a} \cosh ax.$

19. $\int \operatorname{sech}^2 ax \, . \, dx = \frac{1}{a} \tanh ax.$

20. $\int \operatorname{cosech}^2 ax \, . \, dx = -\frac{1}{a} \coth ax.$

21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1} \frac{x}{a} = \log \{x + \sqrt{x^2 + a^2}\}.$

22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \frac{x}{a} = \log \{x + \sqrt{x^2 - a^2}\}.$

23. $\int \frac{dx}{a^2 - x^2} = \frac{1}{a} \tanh^{-1} \frac{x}{a} = \frac{1}{2a} \log \frac{a + x}{a - x}.$

This is indeterminate because $\tan^{-1} \frac{b}{a}$ may have any number of times 2π in it, and indeed the indeterminateness might have been expected as $\epsilon^{2\pi i} = 1$.

Evidently $\cosh x = \cos ix$,

$\sinh x = -i \sin ix$.

sinh is usually pronounced *shin*.

tanh is usually pronounced *tank*.

Prove that if $u = \sinh^{-1} x$ or $x = \frac{1}{2}(\epsilon^u - \epsilon^{-u})$, only positive values of u being taken, then $\epsilon^u = x + \sqrt{1 + x^2}$, and therefore $u = \sinh^{-1} x = \log(x + \sqrt{1 + x^2})$.

Similarly $\cosh^{-1} x = \log(x + \sqrt{x^2 - 1})$,

$$\tanh^{-1} x = \tfrac{1}{2} \log \frac{1 + x}{1 - x},$$

$$\operatorname{sech}^{-1} x = \log\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1}\right),$$

$$\operatorname{cosech}^{-1} x = \log\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}\right).$$

Now compare

$$\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x, \quad \int \frac{dx}{\sqrt{1 + x^2}} = \sinh^{-1} x = \log(x + \sqrt{1 + x^2}).$$

$$\int \frac{dx}{\sqrt{x^2 - 1}} = \cosh^{-1} x = \log(x + \sqrt{x^2 - 1}).$$

$$\int \frac{dx}{1 + x^2} = \tan^{-1} x, \quad \int \frac{dx}{1 - x^2} = \tanh^{-1} x = \tfrac{1}{2} \log \frac{1 + x}{1 - x}.$$

24. $\displaystyle\int \frac{dx}{(x-\alpha)(x-\beta)} = \frac{1}{\alpha-\beta}\log\frac{x-\alpha}{x-\beta}.$

25. $\displaystyle\int \frac{dx}{\sqrt{2ax-x^2}} = \text{vers}^{-1}\frac{x}{a}, \int \frac{dx}{x\sqrt{2ax-a^2}} = \frac{1}{a}\sin^{-1}\frac{x-a}{x}.$

26. $\displaystyle\int \sqrt{(a^2-x^2)}\,.\,dx = \tfrac{1}{2}x\sqrt{a^2-x^2} + \tfrac{1}{2}a^2\sin^{-1}\frac{x}{a}.$

27. $\displaystyle\int \sqrt{x^2+a^2}\,.\,dx = \tfrac{1}{2}x\sqrt{x^2+a^2} + \tfrac{1}{2}a^2\log\{x+\sqrt{x^2+a^2}\}.$

28. $\displaystyle\int \sqrt{x^2-a^2}\,.\,dx = \tfrac{1}{2}x\sqrt{x^2-a^2} - \tfrac{1}{2}a^2\log\{x+\sqrt{x^2-a^2}\}.$

29. $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}} = -\frac{1}{a}\sin^{-1}\frac{a}{x} \text{ or } = \frac{1}{a}\cos^{-1}\frac{a}{x}.$

30. $\displaystyle\int \frac{dx}{x\sqrt{a^2\pm x^2}} = \frac{1}{a}\log\frac{x}{a+\sqrt{a^2\pm x^2}}.$

31. $\displaystyle\int \frac{1}{x}\sqrt{a^2\pm x^2}\,.\,dx = \sqrt{a^2\pm x^2} - a\log\frac{a+\sqrt{a^2\pm x^2}}{x}.$

32. $\displaystyle\int \frac{1}{x}\sqrt{x^2-a^2}\,.\,dx = \sqrt{x^2-a^2} - a\cos^{-1}\frac{a}{x}.$

33. $\displaystyle\int \frac{x\,.\,dx}{\sqrt{a^2\pm x^2}} = \pm\sqrt{a^2\pm x^2}.$

34. $\displaystyle\int \frac{x\,.\,dx}{\sqrt{x^2-a^2}} = \sqrt{x^2-a^2}.$

35. $\displaystyle\int x\sqrt{x^2\pm a^2}\,.\,dx = \tfrac{1}{3}\sqrt{(x^2\pm a^2)^3}.$

36. $\displaystyle\int x\sqrt{a^2-x^2}\,.\,dx = -\tfrac{1}{3}\sqrt{(a^2-x^2)^3}.$

37. $\displaystyle\int \sqrt{2ax-x^2}\,.\,dx = \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x-a}{a}.$

38. $\displaystyle\int \frac{dx}{(x+1)\sqrt{x^2-1}} = \sqrt{\frac{x-1}{x+1}}.$

39. $$\int \frac{dx}{(x-1)\sqrt{x^2-1}} = -\sqrt{\frac{x+1}{x-1}}.$$

40. $$\int \sqrt{\frac{1+x}{1-x}} . dx = \sin^{-1} x - \sqrt{1-x^2}.$$

41. $$\int \sqrt{\frac{x+a}{x+b}} . dx = \sqrt{(x+a)(x+b)} + (a-b)\log(\sqrt{x+a}+\sqrt{x+b}).$$

42. $$\int x^{m-1}(a+bx^n)^{\frac{p}{q}} . dx.$$

1st. If p/q be a positive integer, expand, multiply and integrate each term.

2nd. Assume $a + bx^n = y^q$, and if this fails,

3rd. Assume $ax^{-n} + b = y^q$. This also may fail to give an easy answer.

43. $$\int \sin^{-1} x . dx = x \sin^{-1} x + \sqrt{1-x^2}.$$

44. $$\int x \log x . dx = \frac{x^2}{2}(\log x - \tfrac{1}{2}).$$

45. $$\int x\epsilon^{ax}\, dx = \frac{1}{a}\epsilon^{ax}\left(x - \frac{1}{a}\right).$$

46. $$\int x^n \epsilon^{ax} . dx = \frac{1}{a} x^n \epsilon^{ax} - \frac{n}{a}\int x^{n-1}\epsilon^{ax} . dx.$$

Observe this first example of a formula of reduction to reduce n by successive steps.

47. $$\int \frac{\epsilon^{ax}}{x^m} dx = -\frac{1}{m-1}\frac{\epsilon^{ax}}{x^{m-1}} + \frac{a}{m-1}\int \frac{\epsilon^{ax}}{x^{m-1}} dx.$$

48. $$\int \epsilon^{ax} \log x . dx = \frac{1}{a}\epsilon^{ax}\log x - \frac{1}{a}\int \frac{\epsilon^{ax}}{x} dx.$$

49. $$\int \frac{dx}{\cos x} = \tfrac{1}{2}\log\frac{1+\sin x}{1-\sin x} = \log\left\{\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\}.$$

50. $$\int \frac{dx}{\sin x} = \log\left(\tan\frac{x}{2}\right).$$

51. $$\int \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}} \tan^{-1} \frac{\sqrt{a-b}\tan\frac{x}{2}}{\sqrt{a+b}} \text{ if } a>b,$$

$$= \frac{1}{\sqrt{b^2-a^2}} \log \frac{\sqrt{b-a}\tan\frac{x}{2}+\sqrt{b+a}}{\sqrt{b-a}\tan\frac{x}{2}-\sqrt{b+a}} \text{ if } a<b.$$

52. $$\int \epsilon^{cx} \sin ax \,.\, dx = \frac{\epsilon^{cx}(c\sin ax - a\cos ax)}{a^2+c^2}.$$

53. $$\int \epsilon^{cx} \cos ax \,.\, dx = \frac{\epsilon^{cx}(c\cos ax + a\sin ax)}{a^2+c^2}.$$

54. $$\int \sin^n x \,.\, dx = -\frac{\cos x \,.\, \sin^{n-1} x}{n} + \frac{n-1}{n}\int \sin^{n-2} x \,.\, dx.$$

55. $$\int \frac{dx}{\sin^n x} = -\frac{\cos x}{(n-1)\sin^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2} x}.$$

56. $$\int \epsilon^{ax} \sin^n x \,.\, dx = \frac{\epsilon^{ax}}{a^2+n^2} \sin^{n-1} x\,(a\sin x - n\cos x)$$
$$+ \frac{n(n-1)}{a^2+n^2}\int \epsilon^{ax} \sin^{n-2} x \,.\, dx.$$

57*. $$\int \sin mx \,.\, \sin nx \,.\, dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}.$$

58. $$\int \cos mx \,.\, \cos nx \,.\, dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)}.$$

59. $$\int \sin mx \,.\, \cos nx \,.\, dx = -\frac{\cos(m+n)x}{2(m+n)} - \frac{\cos(m-n)x}{2(m-n)}.$$

60. $$\int \sin^2 nx \,.\, dx = \tfrac{1}{2}x - \frac{1}{4n}\sin 2nx.$$

61. $$\int \cos^2 nx \,.\, dx = \frac{1}{4n}\sin 2nx + \tfrac{1}{2}x.$$

* In integrating any of these products 57 to 61 we must recollect the following formulae:

$$2\sin mx \,.\, \sin nx = \cos(m-n)x - \cos(m+n)x.$$
$$2\cos mx \cos nx = \cos(m-n)x + \cos(m+n)x.$$
$$2\sin mx \cos nx = \sin(m+n)x + \sin(m-n)x.$$
$$\cos 2nx = 2\cos^2 nx - 1 = 1 - 2\sin^2 nx.$$

In the following examples 62 to 67, m and n are supposed to be unequal integers.

62. $\int_0^{\pi \text{ or } 2\pi} \sin mx \,.\, \sin nx \,.\, dx = 0.$

63. $\int_0^{\pi \text{ or } 2\pi} \cos mx \,.\, \cos nx \,.\, dx = 0.$

64. $\int_0^{\pi} \sin^2 nx \,.\, dx = \frac{\pi}{2}, \; \int_0^{\pi} \cos^2 nx \,.\, dx = \frac{\pi}{2}$, if n is an integer.

65. $\int_0^{2\pi} \sin mx \,.\, \cos nx \,.\, dx = 0.$

66. $\int_0^{\pi} \sin mx \,.\, \cos nx \,.\, dx = 0$ if $m - n$ is even.

67. $\int_0^{\pi} \sin mx \,.\, \cos nx \,.\, dx = \frac{m}{m^2 - n^2}$ if $m - n$ is odd.

68. $\int \sin^m x \,.\, \cos x \,.\, dx = \frac{\sin^{m+1} x}{m+1}.$

69. $\int \cos^m x \,.\, \sin x \,.\, dx = -\frac{\cos^{m+1} x}{m+1}.$

Hence any odd power of $\cos x$ or $\sin x$ may be integrated, because we may write it in the form $(1 - \sin^2 x)^n \cos x$ or $(1 - \cos^2 x)^n \sin x$, and if we develope we have terms of the above shapes. Similarly $\sin^p x \,.\, \cos^q x$ may be integrated when either p or q is an odd integer.

70. $\int x^m \sin x \,.\, dx = - x^m \cos x + m \int x^{m-1} \cos x \,.\, dx.$

71. $\int x^m \cos x \,.\, dx = x^m \sin x - m \int x^{m-1} \sin x \,.\, dx.$

72. $\int \frac{\sin x}{x^m} \,.\, dx = -\frac{1}{m-1} \frac{\sin x}{x^{m-1}} + \frac{1}{m-1} \int \frac{\cos x}{x^{m-1}} \, dx.$

73. $\int \frac{\cos x}{x^m} \, dx = -\frac{1}{m-1} \frac{\cos x}{x^{m-1}} - \frac{1}{m-1} \int \frac{\sin x}{x^{m-1}} \, dx.$

74. $\int \tan^n x \,.\, dx = \frac{(\tan x)^{n-1}}{n-1} - \int (\tan x)^{n-2} \,.\, dx.$

75. $\int x^n \sin^{-1} x \,.\, dx = \frac{x^{n+1}\sin^{-1}x}{n+1} - \frac{1}{n+1}\int \frac{x^{n+1}\,dx}{\sqrt{1-x^2}}.$

76. $\int x^n \tan^{-1} x \,.\, dx = \frac{x^{n+1}\tan^{-1}x}{n+1} - \frac{1}{n+1}\int \frac{x^{n+1}\,dx}{1+x^2}.$

77. $\int \frac{dx}{a+bx+cx^2} = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\frac{2cx+b}{\sqrt{4ac-b^2}}$, if $4ac > b^2$,

$= \frac{1}{\sqrt{b^2-4ac}}\log\frac{2cx+b-\sqrt{b^2-4ac}}{2cx+b+\sqrt{b^2-4ac}}$, if $4ac < b^2$,

$= -\frac{2}{2cx+b}$, if $4ac = b^2$.

If $X = a + bx + cx^2$ and $q = 4ac - b^2$, then

78. $\int \frac{dx}{X^2} = \frac{2cx+b}{qX} + \frac{2c}{q}\int\frac{dx}{X}.$

79. $\int \frac{dx}{X^3} = \frac{2cx+b}{q}\left(\frac{1}{2X^2}+\frac{3c}{qX}\right) + \frac{bc^2}{q^2}\int\frac{dx}{X}.$

80. $\int \frac{x\,.\,dx}{X^2} = -\frac{bx+2a}{qX} - \frac{b}{q}\int\frac{dx}{X}.$

81. $\int \frac{dx}{xX} = \frac{1}{2a}\log\frac{x^2}{X} - \frac{b}{2a}\int\frac{dx}{X}.$

82. $\int \frac{dx}{x^2X} = \frac{b}{2a^2}\log\frac{X}{x^2} - \frac{1}{ax} + \left(\frac{b^2}{2a^2}-\frac{c}{a}\right)\int\frac{dx}{X}.$

83. $\int \frac{dx}{\sqrt{(x-a)(b-x)}} = 2\sin^{-1}\sqrt{\frac{x-a}{b-a}}.$

84. $\int \frac{dx}{\sqrt{(a+bx)(\alpha-\beta x)}} = \frac{2}{\sqrt{b\beta}}\sin^{-1}\sqrt{\frac{\beta(a+bx)}{a\beta+b\alpha}}.$

85. $\int \frac{dx}{x\sqrt{x^n-a^2}} = \frac{2}{an}\sec^{-1}\left(\frac{x^{\frac{n}{2}}}{a}\right).$

86. $\int \frac{dx}{x\sqrt{x^n+a^2}} = \frac{1}{an}\log\frac{\sqrt{a^2+x^n}-a}{\sqrt{a^2+x^n}+a}.$

87. $$\int \frac{dx}{\sqrt{a+bx+cx^2}}$$
$$= \frac{1}{\sqrt{c}} \log \left(cx + \frac{b}{2} + \sqrt{c(a+bx+cx^2)} \right).$$

88. $$\int \frac{dx}{\sqrt{a+bx-cx^2}} = \frac{1}{\sqrt{c}} \sin^{-1} \frac{2cx-b}{\sqrt{4ac+b^2}}.$$

89. $\dfrac{(p+gx)}{\sqrt{a+bx+cx^2}}$ may be altered to
$$\frac{g}{2c} \frac{(b+2cx)}{\sqrt{a+bx+cx^2}} + \frac{2pc-gb}{2c} \frac{1}{\sqrt{a+bx+cx^2}},$$
and so integrated.

90. Any integral of the form $\displaystyle\int \frac{P+Q(ax+b)^{p/r}}{R+S(ax+b)^{q/r}} dx$, where P, Q, R and S are rational integral functions of x, can be rationalized by the substitution of $ax+b=v^r$.

91. Any integral of the form $\displaystyle\int \frac{P+Q\sqrt{U}}{R+S\sqrt{U}} dx$, where U is $a+bx+cx^2$, can be rationalized, (1) when b^2-4ac is positive and c negative, by the substitution $\dfrac{\sqrt{-c}\sqrt{U}}{\sqrt{b^2-4ac}} = \dfrac{2y}{1+y^2}$.

(2) When b^2-4ac is positive and c positive by
$$\frac{\sqrt{c}\sqrt{U}}{\sqrt{b^2-4ac}} = \frac{2y}{1-y^2}.$$

(3) When b^2-4ac is negative and a positive by
$$\frac{\sqrt{c}\sqrt{U}}{\sqrt{4ac-b^2}} = \frac{1+y^2}{1-y^2}.$$

If $U=a+bx+cx^2$, $q=4ac-b^2$, $S=\dfrac{4c}{q}$,

92. $$\int \frac{dx}{U\sqrt{U}} = \frac{2(2cx+b)}{q\sqrt{U}}.$$

93. $$\int \frac{dx}{U^n\sqrt{U}} = \frac{2(2cx+b)\sqrt{U}}{(2n-1)qU^n} + \frac{2S(n-1)}{2n-1} \int \frac{dx}{U^{n-1}\sqrt{U}}.$$

94. $$\int \sqrt{U}\,.\,dx = \frac{(2cx+b)\sqrt{U}}{4c} + \frac{1}{2S}\int \frac{dx}{\sqrt{U}}.$$

95. $$\int U\sqrt{U}\,.\,dx = \frac{(2cx+b)\sqrt{U}}{8c}\left(U + \frac{3}{2S}\right) + \frac{3}{8S^2}\int \frac{dx}{\sqrt{U}}.$$

96. $$\int U^n\sqrt{U}\,.\,dx = \frac{(2cx+b)\,U^n\sqrt{U}}{4(n+1)\,c} + \frac{2n+1}{2(n+1)\,S}\int \frac{U^n\,.\,dx}{\sqrt{U}}.$$

97. $$\int \frac{x\,dx}{\sqrt{U}} = \frac{\sqrt{U}}{c} - \frac{b}{2c}\int \frac{dx}{\sqrt{U}}.$$

98. $$\int \frac{dx}{x\sqrt{U}} = -\frac{1}{\sqrt{a}}\log\left\{\frac{\sqrt{U}+\sqrt{a}}{x} + \frac{b}{2\sqrt{a}}\right\} \text{ if } a > 0,$$
$$= \frac{1}{\sqrt{-a}}\sin^{-1}\left(\frac{bx+2a}{x\sqrt{b^2-4ac}}\right) \text{ if } a < 0,$$
$$= \frac{-2\sqrt{U}}{bx} \text{ if } a = 0.$$

99. $$\int \frac{dx}{x^2\sqrt{U}} = -\frac{\sqrt{U}}{ax} - \frac{b}{2a}\int \frac{dx}{x\sqrt{U}}.$$

272. The solutions of many Physical Problems are given in terms of certain well-known definite integrals some of which have been tabulated. The study of these is beyond the scope of this book. I say a few words about **The Gamma Function** which is defined as

$$\int_0^\infty \epsilon^{-x}\,.\,x^{n-1}\,.\,dx = \Gamma(n) \quad \text{..............(1).}$$

By parts, $\int \epsilon^{-x}\,.\,x^n\,.\,dx = -\epsilon^{-x}x^n + n\int \epsilon^{-x}\,.\,x^{n-1}\,.\,dx.$

Putting these between limits it is easy to prove that $-\epsilon^{-x}x^n$ vanishes when $x = 0$ and $x = \infty$.

And therefore $\int_0^\infty \epsilon^{-x}x^n\,.\,dx = n\int_0^\infty \epsilon^{-x}\,.\,x^{n-1}\,.\,dx$ ...(2).

Hence $$\Gamma(n+1)=n\,\Gamma(n)\text{.....................(3)},$$
so that if n is an integer
$$\Gamma(n+1)=1\,.\,2\,.\,3\,.\,4,\ \&c.\ n=\lfloor n\text{............(4)}.$$

Notice that $\lfloor n$ has a meaning *only* when n is an integer, whereas $\Gamma(n)$ is a function of any value of n.

Tables of the values of $\Gamma(n)$ have been calculated, we need not here describe how. The proof that
$$\Gamma(\tfrac{1}{2})=\sqrt{\pi}\text{..........................(5)},$$
as given in most books, is very pretty. The result enables us through (3) to write out $\Gamma(\frac{3}{2})$ or $\Gamma(-\frac{3}{2})$, &c.

A very great number of useful definite integrals can be evaluated in terms of the Gamma Function.

Thus 1. $$\int_0^{\frac{\pi}{2}}\sin^n\theta\,.\,d\theta=\int_0^{\frac{\pi}{2}}\cos^n\theta\,.\,d\theta$$
$$=\tfrac{1}{2}\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)\div\Gamma\left(\frac{n}{2}+1\right).$$

2. $$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx=\int_0^\infty\frac{x^{m+1}\,dx}{(1+x)^{m+n}}=\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$$

3. $$\int_0^\infty \epsilon^{-a^2x^2}\,dx=\frac{1}{2a}\Gamma(\tfrac{1}{2})=\frac{1}{2a}\sqrt{\pi}.$$

4. $$\int_0^\infty x^n\epsilon^{-ax}\,dx=a^{-(n+1)}\,\Gamma(n+1).$$

5. $$\int_0^1 x^m\log\left(\frac{1}{x}\right)^n dx=\frac{\Gamma(n+1)}{(m+1)^{n+1}}.$$

6. $$\int_0^\infty\frac{\epsilon^{-y}\,.\,dy}{2a\sqrt{y}}=\frac{1}{2a}\Gamma(\tfrac{1}{2}).$$

7. $$\tfrac{1}{2}\int_0^1 y^{\frac{1}{2}l-1}(1-y)^{m-1}\,dy=\Gamma\left(\frac{l}{2}\right)\Gamma(m)\div 2\Gamma\left(\frac{l}{2}+m\right).$$

8. $$\int_0^{\frac{\pi}{2}}\sin^p\theta\cos^q\theta\,.\,d\theta=\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q}{2}+1\right)}.$$

APPENDIX.

THE following notes are intended to be read in connection with the text on the page whose number appears before the note. The exact position on the page is indicated by a †.

Page 3. The ordinary propositions in Geometry ought to be *illustrated* by actual drawing. The best sign of the health of our race is shown in this, that for two generations the average British boy has been taught Euclid the mind destroying, and he has not deteriorated. Euclid's proofs are seemingly logical; advanced students know that this is only in appearance. Even if they were logical the Euclidian Philosophy ought only to be taught to men who have been Senior Wranglers. 95 per cent. of the schoolboys, whose lives it makes miserable, are as little capable of taking an interest in abstract reasoning as the other five per cent. are of original thought.

Page 43. There is a much more accurate method for finding $\frac{dp}{dt}$ described in my book on the Steam Engine.

Page 48. This rule is not to be used for values of x greater than 16.

Page 63. In Art. 39 I have given the flue investigations as usually given, but prefer the following method, for I have for some time had reason to believe that in a tube of section A, perimeter P, if W lb. of gases flow per second, the loss of heat per second per unit area of tube is proportional to

$$v\theta/t,$$

if t is the absolute temperature of the gases and v the velocity of the gases. Now $v \propto Wt \div A$, so that the loss of heat per second is proportional to

$$W\theta/A.$$

Proceeding as before

$$-W.d\theta = CW\theta.dS/A,$$

or

$$-A\frac{d\theta}{\theta} = C.dS,$$

or

$$-A\log\theta + c = CS,$$

where c and C are constants. As before, this leads to

$$c = A\log\theta_1, \quad S = \frac{A}{C}\log\frac{\theta_1}{\theta}.$$

Whole $S = \frac{A}{C}\log\frac{\theta_1}{\theta_2}$, so that the efficiency becomes

$$E = 1 - e^{-CS/A}.$$

Now if it is a tube of perimeter P and length l, S/A becomes Pl/A or l/m where m is known as the hydraulic mean depth of the flue, so that

$$E = 1 - \epsilon^{-Cl/m}.$$

This makes the efficiency to be independent of the quantity of stuff flowing, and within reasonable limits I believe that this is true on the assumption of extremely good circulation on the water side. This notion, and experiments illustrating it, were published in 1874 by Prof. Osborne Reynolds before the Manchester Philosophical Society.

Mr Stanton has recently (*Phil. Trans.* 1897) published experiments which show that we have in this a principle which ought to lead to remarkable reductions in the weights of boilers and surface condensers; using extremely rapid circulation and fine tubes.

Page 95. It ought not to be necessary to say here that the compressive stress at any point in the section of a beam such as $ACAC$, fig. 47, is $\frac{M}{I}z$, if z is the distance (say JH) of a point on the compression side of the neutral line AA from the neutral line. The neutral line passes through the centre of the section. If z is negative the stress is a tensile stress. The greatest stresses occur where z is greatest. Beams of uniform strength are those in which the same greatest stress occurs in every section.

Page 111. Mr George Wilson (*Proc. Royal Soc.*, 1897) describes a method of solving the most general problems in continuous beams which is simpler than any other. Let there be supports at A, B, C, D, E. (1) Imagine no supports except A and E, and find the deflections at B, C and D. Now assume only an upward load of any amount at B, and find the upward deflections at B, C, and D. Do the same for C and D. These answers enable us to calculate the required upward loads at B, C and D which will just bring these points to their proper levels.

Page 139. For beginners this is the end of Chap. I.

Page 146. In all cases then,

$$dH = k\,.\,dt + t\left(\frac{dp}{dt}\right)dv \qquad \text{(23*)}.$$

Exercise 1. By means of (23) express K, l, L, P and V in terms of k and write out the most general form of equation (3) in terms of k. Show that among many other interesting statements we have what Maxwell calls the four Thermodynamical relations,

$$\left(\frac{dv}{dt}\right) = -\left(\frac{d\phi}{dp}\right)_t ; \quad \left(\frac{dv}{d\phi}\right)_p = \left(\frac{dt}{dp}\right)_\phi ; \quad \left(\frac{dp}{dt}\right) = \left(\frac{d\phi}{dv}\right)_t ; \quad \left(\frac{dp}{d\phi}\right)_v = -\left(\frac{dt}{dv}\right)_\phi .$$

Exercise 2. Prove that in fig. 55,

$$\text{Area } ABCD = AE\,.\,AF, \; = AU\,.\,AJ = AG\,.\,AM = AQ\,.\,AR,$$

and show that these are the above four relations.

Exercise 3. Show by using (23) with (7), (8), (11) and (14) that for any substance

$$L=-t\left(\frac{dv}{dt}\right),\ k=K-t\left(\frac{dv}{dt}\right)\left(\frac{dp}{dt}\right),\ V=K\left(\frac{dt}{dv}\right),\ P=k\left(\frac{dt}{dp}\right),$$

and that (20) becomes

$$\left(\frac{dk}{dv}\right)_t=t\,\frac{d^2p}{dt^2},$$

so that

$$k=k_0+t\int\left(\frac{d^2p}{dt^2}\right)dv,$$

where k_0 is a function of temperature only.

Dividing $dH=K.dt+L.dp$ by t and stating that it is a complete differential, show that we are led to

$$K=K_0-t\int\frac{d^2v}{dt^2}.dp,$$

where K_0 is a function of temperature only.

Page 152. Another way is merely to recognize (8) as being the same as

$$\delta H=k.\delta t+l.\delta v,$$

for $l=t\left(\frac{dp}{dt}\right)$, and when a pound of stuff of volume s_1 receives $\delta H=L$ at constant temperature (or $\delta t=0$) in increasing to the volume s_2 (so that $\delta v=s_2-s_1$) we have, since dp/dt is independent of v,

$$\delta H=L=0+t.\frac{dp}{dt}.(s_2-s_1).$$

Page 188. Sine functions of the time related to one another by linear operators such as $a+b\theta+c\theta^2+\text{etc.}+e\theta^{-1}+f\theta^{-2}+\text{etc.}$, where θ means $\frac{d}{dt}$, are represented by and dealt with as vectors in the manner here described. Representing an electromotive force and a current in this way, the scalar product means Power. Dr Sumpner has shown (*Proc. Roy. Soc.*, May 1897) that in many important practical problems more complicated kinds of periodic functions may be dealt with by the Vector Method.

Page 190. The symbol $\tan^{-1}$ means "an angle whose tangent is."

Page 195. To understand how we develope a given function in a Fourier Series, it is necessary to notice some of the results of Art. 109, very important for other reasons; indeed, I may say, all-important to electrical engineers.

Page 195. Article 126 should be considered as displaced so as to come immediately before Art. 141, p. 210.

Page 202. The problem of Art. 125 is here continued.

Page 208. The beginner is informed that Σ means "the sum of all such terms as may be written out, writing 1 for s, 2 for s, 3 for s, and so on."

Page 209. See Ex. 23, p. 184.

Page 213. The student ought to alter from v to C or to Q in (9) as an exercise.

Page 213. See Art. 152.

Page 241. Remember that the *effective* value of $a \sin(nt+e)$ is $a \div \sqrt{2}$.

Page 254. Here again a student needs a numerical Example.

Page 256. After copper *read* "and their insulations."

Page 259. Then

$$e_1+e_2=E[\sin(nt+a)+\sin(nt-a)]=2E\cos a\sin nt.$$

(*See* Art. 109.)

Page 266. All other statements about this subject that I have seen are of infantine simplicity, but utterly wrong.

Page 269. Insert "and in consequence."

Page 278. In the same manner show that if

$$y=a^x,\ \frac{dy}{dx}=a^x\log a.$$

Page 281. See the eighth fundamental case, Art. 215.

Page 299. See (1) Note to Art. 21.

Page 301. Article 225 should be considered as displaced so as to precede Art. 222.

Page 305. See Ex. 8, Art. 99.

Page 309. Or (3).

Page 310. Art. 225 should be read before Art. 222.

Page 331. I have taken an approximate law for h and so greatly shortened the work.

Page 359. **Viscosity.** All the fluid in one plane layer moves with the velocity v; the fluid in a parallel plane layer at the distance δx moves with the velocity $v+\delta v$ in the same direction; the tangential force per unit area necessary to maintain the motion is $\mu\frac{\delta v}{\delta x}$ or $\mu\frac{dv}{dx}$, where μ is the viscosity.

Example 1. A circular tube is filled with fluid, the velocity v at any point whose distance from the axis is r being parallel to the axis. Consider the equilibrium of the stuff contained between the cylindric spaces of radii r and $r+\delta r$ of unit length parallel to the axis. The

tangential force on the inner surface is $2\pi r\mu \frac{dv}{dr}$ and on the outer surface it is what this becomes when r is changed to $r+\delta r$ or $2\pi\mu \frac{d}{dr}\left(r\frac{dv}{dr}\right)\delta r$, the difference of pressure between the ends gives us a force $-2\pi r \frac{dp}{dx}\delta r$ if x is measured parallel to the axis. The mass of the stuff is $2\pi r \,.\, \delta r \,.\, m$ if δr is very small and if m is the mass per unit volume; its acceleration is $\frac{dv}{dt}$ if t is time; and hence, equating force to mass $\times$ acceleration and dividing by $2\pi\mu\delta r$,

$$\frac{d}{dr}\left(r\frac{dv}{dr}\right)+\frac{r}{\mu}\frac{dp}{dx}=\frac{r\,.\,m}{\mu}\frac{dv}{dt}.$$

Example 2. Let $\frac{dp}{dx}$ be constant; say that we have a change of pressure P in the length l so that $\frac{dp}{dx}=\frac{P}{l}$. Let a state of steady flow have been reached so that $\frac{dv}{dt}=0$, then

$$\frac{d}{dr}\left(r\frac{dv}{dr}\right)+\frac{r}{\mu}\frac{P}{l}=0.$$

If $r\frac{dv}{dr}$ be called u and if $P/l\mu$ be called $2a$, then $\frac{du}{dr}+2ar=0$, so that $du+2ar\,.\,dr=0$, or $u+ar^2=$ constant c.

$$r\frac{dv}{dr}+ar^2=c, \text{ or } \frac{dv}{dr}+ar=\frac{c}{r} \quad\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$dv+\left(ar-\frac{c}{r}\right)dr=0,$$

$$v+\tfrac{1}{2}ar^2+\tfrac{1}{2}\frac{c}{r^2}=C \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Evidently as there is no tangential force where $r=0$, $\frac{dv}{dr}=0$ there, c must be 0. Hence

$$v+\tfrac{1}{2}ar^2=C\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Now $v=0$ where $r=r_0$, the outer radius of the fluid, and hence (2) becomes $v=\frac{1}{2}a\,(r_0^2-r^2)$.

The volume of fluid per second passing any section is

$$2\pi\int_0^{r_0} rv\,.\,dr=\frac{\pi}{4}ar_0^4=\pi r_0^4P/8l\mu.$$

This enables us to calculate the viscosity of a fluid passing through a cylindrical tube if we know the rate of flow for a given difference of pressure.

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